

# Some cosmological and astrophysical aspects of modified gravity theories

Álvaro de la Cruz Dombriz<sup>1</sup>

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Dr. Antonio Dobado González and Dr. Antonio López Maroto



Universidad Complutense de Madrid  
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<sup>1</sup>dombriz@fis.ucm.es



*A Carmen y Julio Dombriz,*

cuyos ojos  
no alcanzaron este día,  
pero cuyas palabras  
me enseñaron a hablar.

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*Beauty is truth, truth beauty,  
that is all Ye know on earth,  
and all ye need to know.*

John Keats,  
*Ode on a Grecian Urn*



# Contents

<b>Preface</b>	<b>1</b>
<b>1 Introduction to modified gravity theories</b>	<b>5</b>
1.1 Motivation . . . . .	5
1.2 Generalities . . . . .	8
1.3 Modified Einstein equations . . . . .	10
1.4 Equivalence with Brans-Dicke theories . . . . .	12
1.5 Geometrical results . . . . .	14
1.5.1 Vacuum solutions . . . . .	14
1.5.2 Some EH solutions reproduced by $f(R)$ theories . . . . .	15
1.6 Constraints on $f(R)$ theories to ensure viability . . . . .	17
1.7 Brane-world theories . . . . .	18
1.8 Excitations in brane worlds: branons . . . . .	19
1.9 Brane-skyrmions . . . . .	22
1.9.1 Static brane-skyrmions . . . . .	23
1.10 Gravitational signatures at the LHC . . . . .	25
<b>2 Dark energy in <math>f(R)</math> theories</b>	<b>27</b>
2.1 Introduction . . . . .	27
2.2 Standard Einstein equations in a FLRW universe . . . . .	28

2.3	Modified Einstein equations in a FLRW universe . . . . .	30
2.4	Cosmological viability for $f(R)$ dark energy models . . . . .	31
2.4.1	Critical points and stability . . . . .	31
2.4.2	Classification of $f(R)$ models . . . . .	31
2.5	Some cosmologically viable $f(R)$ models . . . . .	33
2.6	$f(R)$ with no cosmological constant . . . . .	35
2.6.1	Cosmological evolution in $\Lambda$ CDM model . . . . .	36
2.6.2	$f(R)$ case with no cosmological constant . . . . .	36
2.7	Effective fluid description of $f(R)$ gravities . . . . .	41
2.7.1	Some examples . . . . .	43
2.8	Conclusions . . . . .	44
<b>3</b>	<b>Cosmological perturbations in <math>f(R)</math> theories</b>	<b>45</b>
3.1	Introduction . . . . .	45
3.2	Theory of cosmological perturbations . . . . .	46
3.2.1	Generalities . . . . .	46
3.2.2	Gauge-invariant variables and gauge choice . . . . .	48
3.2.3	Equations for cosmological perturbations in EH gravity . . . . .	50
3.3	Cosmological perturbations in $f(R)$ theories . . . . .	53
3.3.1	Perturbed Einstein equations in $f(R)$ theories . . . . .	53
3.3.2	Equation for density perturbations in $f(R)$ theories . . . . .	55
3.3.3	Evolution of sub-Hubble modes and the quasi-static approximation	57
3.3.4	Some proposed models . . . . .	60
3.4	A viable $f(R)$ model different from $\Lambda$ CDM? . . . . .	62
3.5	Conclusions . . . . .	66
<b>4</b>	<b>Black holes in <math>f(R)</math> theories</b>	<b>69</b>

4.1	Introduction . . . . .	69
4.2	Constant curvature black-hole solutions . . . . .	71
4.3	Perturbative results . . . . .	74
4.3.1	General expression to arbitrary order for constant curvature . . . .	77
4.4	Black-hole thermodynamics . . . . .	79
4.5	Particular examples . . . . .	83
4.6	Figures for thermodynamical regions . . . . .	86
4.7	Conclusions . . . . .	92
<b>5</b>	<b>Brane-skyrmions and the CMB cold spot</b>	<b>93</b>
5.1	Introduction . . . . .	93
5.2	Spherically symmetric brane-skyrmions . . . . .	94
5.3	Cold spot in WMAP data . . . . .	98
5.4	Cold spot as a cosmic texture . . . . .	99
5.5	Physical interpretation of the results and involved scales . . . . .	100
5.5.1	Brane-skyrmions abundance . . . . .	101
5.6	Future prospects and conclusions . . . . .	102
<b>6</b>	<b>Conclusions and prospects</b>	<b>103</b>
<b>A</b>	<b>Coefficients in <math>f(R)</math> cosmological perturbations</b>	<b>107</b>
A.1	Appendix <b>I</b> : $\alpha's$ and $\beta's$ coefficients . . . . .	107
A.2	Appendix <b>II</b> : $c's$ coefficients . . . . .	109
	<b>Bibliography</b>	<b>111</b>



# Preface

The twentieth century witnessed the development of both gravitation and cosmology as modern scientific disciplines subjected to observations. These observations have been performed through terrestrial particle detection devices, telescopes and satellites that allow to verify theoretical predictions and to rule out proposed theoretical models. With the turning of the new century, called to be the century of precision cosmology, new perspectives have been unveiled with recent experiments such as WMAP, PLANCK or SDSS. These last experiments are able to determine with higher and higher accuracy the features of the Cosmic Microwave Background (CMB), the distribution of large scale structures and the fundamental cosmological parameters which describe our universe on the largest scales. Despite the improvements in the observational side, a fundamental gravitational theory, which is renormalizable from a quantum field theory point of view and applicable to arbitrary scales, from micro-gravitational tests, passing through solar system tests, to cosmological scales, is still lacking.

General relativity, in spite of being the most successful gravitational theory in the last one hundred years, has left some of these problems without satisfactory answer. Although within the string theory paradigm it would be possible to find a consistent quantum theory of gravity, this is not the case of general relativity which turns out to be nonrenormalizable as a perturbative field theory. Moreover, if this theory is used to construct the standard cosmological model, where the fluid content is given by standard matter and radiation, it cannot account for the observed accelerated expansion of the universe on sufficiently large scales. In fact, it needs to be supplemented by some dark energy contribution to accommodate this accelerated regime. On the other hand, general relativity with gravitating luminous matter cannot account either for the observed rotation curves of galaxies. A dark matter contribution needs to be introduced to reconcile data with theoretical predictions within this paradigm.

Instead of adding new elements in the cosmological content, which try to accommodate observations with general relativity, those problems might show that the theoretical framework in cosmology should be enlarged by alternative gravity theories. This thesis will try to contribute to the understanding of those still open issues by considering two recently proposed alternative and complementary theories to general relativity. We shall

consider some relevant aspects of those models related to recent experimental results.

The present work is organized in the way that follows: First, we will briefly introduce in Chapter 1 some modified gravity theories and their corresponding formalisms. In this chapter special attention will be paid to  $f(R)$  gravities by summarizing the main features of this paradigm in the metric formalism. Then some geometrical results for  $f(R)$  theories and both cosmological and gravitational constraints usually imposed over such functions will be provided. Other alternative modified gravities, the brane worlds, will then be reviewed. Here we shall introduce both the notion of brane excitations, the branons, and some topologically nontrivial solutions, the brane-skyrmions. We shall finish the chapter by providing some insight about the possibility of mini black holes detection in the Large Hadron Collider (LHC) as a signature for the validity of these modified gravity theories.

The second chapter will deal with  $f(R)$  theories which try to provide a cosmological acceleration mechanism with no need for introducing any extra dark energy contribution in the cosmological components. To do so, we shall use some reconstruction procedures which start either from a given solution of the cosmological scale factor for an homogeneous and isotropic metric or from an effective equation of state. In particular, those  $f(R)$  functions able to mimic Einstein-Hilbert plus cosmological constant solutions will be obtained. In this realm,  $f(R)$  theories will be shown to be able to mimic the cosmological evolution generated by any perfect fluid with constant equation of state.

Then the third chapter will be devoted to the computation of cosmological perturbations for  $f(R)$  theories. Since in Chapter 2 the modified Einstein equations will have been studied as background equations, it is quite natural when modifying general relativity by  $f(R)$  models, to ask about the first order perturbed equations for these theories and what consequences in the growing of these perturbations may appear. This is the *leitmotiv* in this chapter. Throughout it, special attention will be paid to the possibility of obtaining a completely general differential equation for the evolution of perturbations and its particularization for the so-called sub-Hubble scales will be explicitly shown. The mentioned differential equation in those scales will be shown to be very useful to understand the regime validity of some approximations widely accepted in the literature and to rule out that some proposed  $f(R)$  models could be cosmologically viable.

The introduction of modified gravity theories, with or without extra dimensions, may lead to the existence of new solutions with respect to those of general relativity. In that sense, the research about spherically symmetric solutions is of particular interest. For instance, it may shed some light on the number of extra dimensions, the fundamental scale of gravity or the required restrictions to be imposed over the parameters of those theories. The possible detection of mini black holes at the LHC in the coming years will be a turning point to discover certain properties of the underlying gravity theory. For this reason, chapters 4 and 5 will be devoted to the study of spherical solutions in extra dimensions theories. In particular, spherically symmetric and static black-hole solutions coming from

$f(R)$  theories in an arbitrary number of dimensions will be studied in Chapter 4 whereas Chapter 5 will be focused on studying a particular topologically nontrivial solution with spherical symmetry appearing in brane-world models – different from the well-known black-hole solutions – the so-called brane-skyrmions.

Hence, in the fourth chapter we shall focus on the study of black holes in  $f(R)$  gravity theories in an arbitrary number of dimensions. We shall concentrate on the existence of black-hole solutions and we shall study which will be their inherited or different features with respect to those in general relativity. With this purpose we shall study constant curvature solutions for  $f(R)$  theories as well as perturbative solutions around the standard Schwarzschild-anti-de Sitter geometry. An important part of this chapter will be then devoted to the thermodynamics of Schwarzschild-anti-de Sitter black holes in  $f(R)$  theories. This research will prove that for  $f(R)$  gravities, there exists a thermodynamical viability condition which is related to one of the conditions which ensure gravitational viability for  $f(R)$  models.

In the fifth chapter we will thoroughly study other kind of spherically symmetric solutions in brane-world theories that are not black holes. These solutions, the brane-skyrmions, are topologically nontrivial configurations arising in the presence of these extra dimensions theories. In this context, the recent claim of detection of an unexpected feature in the CMB, referred to as the cold spot, will be explained as a topological defect on the brane. After performing some calculations, it will be shown that results obtained are in complete agreement with those in the literature that tried to explain that cold spot as a texture of a non-linear sigma model. The physical interpretation of these results and future prospects will finish this chapter.

At the end of each chapter, we shall include the corresponding conclusions. These conclusions are summarized all together in the sixth chapter, which is followed by an appendix where more detailed formulae for the calculations performed in the third chapter are shown.



# Chapter 1

## Introduction to modified gravity theories

### 1.1 Motivation

From its very beginning, it was questioned whether general relativity (GR) was the unique correct theory among other theories for gravitation. Thus for instance Weyl in [1] and Edington in [2] included higher order invariants in the gravitational action. Those attempts were neither experimentally nor theoretically motivated, but it was soon proved that the Einstein-Hilbert (EH) action was not renormalizable and therefore could not be conventionally quantized. In fact, this action needs to be supplemented by higher order terms in order for the resultant theory to be one-loop renormalizable [3, 4]. More recent research has shown that when quantum loop corrections in field theory or higher order corrections in the low energy string dynamics are considered, the effective low energy gravitational action includes higher order curvature invariants [5, 6, 7].

Such results encouraged the interest in higher order gravity theories, i.e., modifications of the EH gravitational action which include higher order curvature invariants. Nonetheless, those new added contributions were thought to be relevant only in very strong gravity regimes, such as at scales close to the Planck scale and therefore in the early universe or near black hole singularities. However, these corrections were not expected to affect gravitational phenomenology neither at low curvature nor at low energy regimes, and therefore they were assumed to be negligible at large scales such as those involved in the late universe evolution.

Very recent evidence coming from both astrophysics and cosmology have revealed the unexpected accelerated expansion of the universe. Different data from type Ia supernovae (SNIa) surveys [8, 9, 10], large structure formation and delicate measurements of the CMB

anisotropies, particularly those from the Wilkinson Microwave Anisotropy Probe (WMAP) [11], have concluded that our universe is expanding at an increasing rate. This fact sets the very urgent problem of finding the cause for this speed-up since standard GR with ordinary matter and radiation is not able to do so. Usual explanations for this fact have been categorized to belong to one of the following three classes:

1. The first type of explanations reconciles this acceleration with GR by invoking a strange cosmic fluid, dark energy (DE) (see [12] and references therein), with a state equation relating its pressure and energy density in the following way

$$P_{\text{DE}} = \omega_{\text{DE}} \rho_{\text{DE}} \quad (1.1)$$

where  $\omega_{\text{DE}} < -1/3$  is required to provide acceleration in the usual Einstein equations as is described in Section 2.2. This state equation shows that the DE fluid has a large negative pressure. For the particular case  $\omega_{\text{DE}} = -1$ , this fluid behaves just as a cosmological constant  $\Lambda$ . Within this approach of DE in the form of a cosmological constant, recent data obtained by WMAP [11] provide the following cosmological content distribution: 4.6% corresponds to ordinary baryonic matter, 22.7% to cold dark matter and 72.9% to DE. This is the so-called concordance or  $\Lambda$ -Cold Dark Matter model ( $\Lambda$ CDM) which is supplemented with some inflation mechanism usually through some scalar field, the inflaton. The main problem of this kind of description is that the fitted  $\Lambda$  value seems to be about 55 orders of magnitude smaller than the expected vacuum energy of matter fields, this is the so-called *cosmological constant problem*. From a more philosophical point of view, the DE description also presents the so-called *coincidence problem*. This problem wonders why the DE and matter densities are so close in order of magnitudes precisely in these days, i.e. in the present cosmological era, even though for both the cosmological past and future that is not the case. This kind of problems comes to claim that the  $\Lambda$ CDM model could be regarded as an empirical fit to data with a poorly motivated gravitational theory behind and therefore, it should be considered as a phenomenological approach of the underlying correct cosmological theory.

2. The second type of explanations consider a dynamical DE by introducing a new scalar field. They are the so-called quintessence theories. The theories which introduce an extra scalar field in the gravitational sector of the action are usually referred to as scalar-tensor theories. Some very interesting subcases of such theories are the so-called Brans-Dicke theories which are going to be explained in detail in the Section 1.4.
3. Finally the third one consists of trying to explain the cosmic acceleration as a consequence of new gravitational physics [13, 14]. For instance, modifications

to the EH gravitational action have been widely considered in the literature [15, 16, 17, 18, 19, 20]. More recently, vector-tensor theories of gravity and the electromagnetic field itself have also been proposed as compelling DE candidates [21].

Some of those theories add higher or lower powers of the scalar curvature, the Riemann and Ricci tensors or their derivatives [22]. Lovelock theories and  $f(R)$  gravity theories are some examples of these attempts. In recent years, some  $f(R)$  proposals have even tried to reconcile dark matter through a gravitational sector modification [23] or to explain both the current cosmic speed-up and early inflation simultaneously [24]. The core of Chapter 2 will thoroughly deal with some attempts of  $f(R)$  theories to circumvent the necessity of introducing DE to explain the cosmic acceleration.

On the other hand, other open issues in the Standard Model (SM) of elementary particles, namely the *hierarchy problem*, could also be related to the fundamental gravity theory. Thus, this problem appears in the renormalization procedure in theories containing scalar fields. In such theories the renormalized scalar masses are expected to be given by the cut-off of the theory, i.e., the Planck scale. Therefore an extreme fine tuning is required in order to get the expected mass for scalars, in particular the Higgs mass. If on the contrary the fundamental scale of gravitation is close to the electroweak scale, the corresponding cut-off would be of the same order as the expected Higgs mass and an extreme fine tuning would not be required.

With the aim of solving this problem, large extra dimensions theories have recently been considered. Unlike ancient Kaluza-Klein theories, with compactified Planck scale size extra dimensions, recent brane-world models may contain much larger extra dimensions. In order to avoid the presence of Kaluza-Klein towers of copies of SM particles with similar masses, these models restrict SM particles to propagate on the brane, whereas only gravity can propagate in the whole bulk space. In this way, the fundamental gravity scale can be reduced to the electroweak scale and the gauge *hierarchy problem* is avoided.

Brane-world (BW) theories may also explain the observed accelerated expansion of the universe [19] and as will be shown in Section 1.8, they present excitations which can produce weakly interacting massive particles (WIMPs), which are natural candidates for the observed dark matter [25]. Let us finally remark that such modified extra dimensions gravity theories, as will be explained in Section 1.10, may give rise to the production of black holes (BHs) of little size at the LHC whose eventual detection may give valuable information about the dimensionality of space-time.

In the following sections of this chapter we shall deal with different aspects of the already mentioned modified gravity theories, both  $f(R)$  theories and brane-world theories: in Section 1.2 we shall present some generalities about the formalism which will be used throughout the thesis. Thus in Section 1.3, the modified Einstein equations derived

from  $f(R)$  theories in the metric formalism will be presented. Then in Section 1.4, the equivalence of such theories with Brans-Dicke theories will be briefly sketched. Some geometrical results for  $f(R)$  gravities which were originally published in [26] will be presented in Section 1.5. They deal with constant curvature solutions and analytical conditions to reproduce Einstein's equations for the EH action with or without cosmological constant. Concerning BW theories, the main concepts for those models are analyzed in Section 1.7, while a study of BW excitations, called branons, is presented in Section 1.8. Topologically nontrivial configurations, called skyrmions are introduced in Section 1.9 and some gravitational consequences of those theories will be summarized in the final Section 1.10.

## 1.2 Generalities

The gravitational action for GR in an arbitrary number of dimensions  $D$  is given by the so-called EH action

$$S_{\text{EH}} = \frac{1}{2\kappa} \int d^D x \sqrt{|g|} R. \quad (1.2)$$

Here,  $\kappa \equiv 8\pi G_D$  where  $G_D \equiv M_D^{2-D}$  holds for the  $D$ -dimensional gravitational constant, with  $M_D$  the gravitational fundamental scale,  $g$  is the metric determinant and  $R$  is the Ricci scalar defined from the metric tensor.

With the aim of modifying the EH action, gravitational action for  $f(R)$  theories, considered as generalizations of GR, may be written as

$$S_G = \frac{1}{2\kappa} \int d^D x \sqrt{|g|} (R + f(R)). \quad (1.3)$$

From either actions given in equations (1.2) or (1.3), the field equations, giving rise to the so-called standard and modified Einstein equations respectively, can be derived by using different variational principles. Two such variational principles have been mainly considered in the literature: on the one hand, the standard metric formalism considers that the connection is metric dependent and therefore the only present fields in the gravitational sector are those coming from the metric tensor. On the other hand, there exists the so-called Palatini variational principle where metric and connection are assumed to be independent fields. In this case the action is varied with respect to both of them. Whereas for an action linear in  $R$  such as that in expression (1.2) both formalisms lead to the same field equations, this is no longer true for nonlinear gravity theories (see [27] for an exhaustive review on nonmetric formalisms). In this thesis, we shall restrict ourselves to the metric formalism. For that purpose, we shall assume that the connection is the usual

Levi-Civita connection given by

$$\Gamma_{\mu\nu}^{\alpha} \equiv \frac{1}{2}g^{\alpha\gamma} \left( \frac{\partial g_{\gamma\nu}}{\partial x^{\mu}} + \frac{\partial g_{\mu\gamma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\gamma}} \right) \quad (1.4)$$

where, as in the rest of the work, Einstein's convention for implicit summation is assumed.

At this stage, let us point out that the convention to be used for the metric signature will be  $(+, -, \dots, -)$ , i.e., positive sign for temporal coordinate whereas negative sign for spatial ones. With respect to the Riemann tensor definition, our conventions will be

$$R^{\mu}_{\nu\alpha\beta} \equiv \frac{\partial \Gamma_{\nu\alpha}^{\mu}}{\partial x^{\beta}} - \frac{\partial \Gamma_{\nu\beta}^{\mu}}{\partial x^{\alpha}} + \Gamma_{\sigma\beta}^{\mu}\Gamma_{\nu\alpha}^{\sigma} - \Gamma_{\sigma\alpha}^{\mu}\Gamma_{\nu\beta}^{\sigma}. \quad (1.5)$$

From expression (1.5), the corresponding Ricci tensor and scalar curvature are obtained straightforwardly and they read respectively as follows

$$R_{\mu\nu} \equiv R^{\alpha}_{\mu\alpha\nu} ; R \equiv R^{\alpha}_{\alpha}. \quad (1.6)$$

In addition to the already explained gravitational sector, the energy content may be introduced in the cosmological content through energy-momentum tensors, which will describe the different components such as dust matter, radiation, dark matter, etc. which are present in the cosmological content of the universe. For each different type of fluid content ( $\alpha$ ), assumed from now on to behave as a perfect fluid, the corresponding energy-momentum tensor is given by

$$T_{\mu\nu}^{(\alpha)} = (P_{\alpha} + \rho_{\alpha})u_{\mu}^{(\alpha)}u_{\nu}^{(\alpha)} - P_{\alpha}g_{\mu\nu} \quad (1.7)$$

where  $P_{\alpha}$ ,  $\rho_{\alpha}$  and  $u^{\mu(\alpha)}$  are the pressure, energy density and 4-velocity of the  $\alpha$  component respectively. Therefore the total energy-momentum tensor will be nothing but

$$T_{\mu\nu} \equiv \sum_{\alpha} T_{\mu\nu}^{(\alpha)} \quad (1.8)$$

for all possible fluid contributions. The most usual approach is to consider barotropic fluids where  $P_{\alpha} = P_{\alpha}(\rho_{\alpha})$  and very often the relation between these two quantities is linear through an equation of state

$$P_{\alpha} = \omega_{\alpha}\rho_{\alpha} \quad (1.9)$$

where for instance  $\omega_{\alpha} = -1, 0, 1/3$  if cosmological constant, dust matter or radiation are the considered fluids respectively. DE fluids with constant equation of state are given by the condition  $\omega_{\text{DE}} < -1/3$  whereas phantom candidates for DE obey  $\omega_{\text{DE}} < -1$ . In our approach to modify GR, to be rigorously implemented in Chapter 2, DE will appear as a modification of the gravitational sector itself so no DE component will be explicitly included in the content expressed by the summation (1.8). In this case the cosmic acceleration will be a consequence of the modification of the gravitational action by the presence of a  $f(R)$  term. Let us finish this section by mentioning that each fluid component is assumed to be conserved separately since no interaction among fluids is considered. This fact also implies the conservation of the total energy-momentum tensor straightforwardly.

### 1.3 Modified Einstein equations

Now that the previous generalities have been presented, the modified Einstein equations in the metric formalism for  $f(R)$  gravity theories may be found by performing variations of the gravitational action (1.3) with respect to the metric and equating the result to minus the energy-momentum tensor times  $\kappa$  providing the following equations:

$$(1 + f_R)R_{\mu\nu} - \frac{1}{2}(R + f(R))g_{\mu\nu} + \mathcal{D}_{\mu\nu}f_R = -\kappa T_{\mu\nu} \quad (1.10)$$

where  $f_R \equiv df(R)/dR$  and

$$\mathcal{D}_{\mu\nu} \equiv \nabla_\mu \nabla_\nu - g_{\mu\nu} \square \quad (1.11)$$

with  $\square \equiv \nabla_\alpha \nabla^\alpha$  and  $\nabla$  is the usual covariant derivative.

Taking the trace of the equation (1.10) we get:

$$R(1 + f_R) - \frac{D}{2}(R + f(R)) + (1 - D)\square f_R = -\kappa T \quad (1.12)$$

which provides a differential relation between  $R$  and  $T$  unlike GR where this relation is just algebraic. An interesting point to stress at this stage is that in general, vacuum solutions, i.e.  $T_{\mu\nu} \equiv 0$ , do not imply straightforwardly  $R = 0$  solutions.

By computing the covariant derivative of (1.10), it is found that the l.h.s. of those equations vanishes identically, so the covariant derivative for the r.h.s. of equations (1.10) must obey the conservation equations

$$\nabla_\mu T^\mu_\nu = 0 \quad (1.13)$$

where this identity does not depend explicitly on  $f(R)$  but only on the energy-momentum tensor components and metric tensor elements.

Two particular simple choices for  $f(R)$  may be considered in the equations (1.10):

1.  $f(R) \equiv 0$ , which allows to recover the standard Einstein equations without cosmological constant, i.e.,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = -\kappa T_{\mu\nu} \quad (1.14)$$

where the conservation equations (1.13) still hold.

2. A second simple choice would be  $f(R) \equiv -(D-2)\Lambda_D$ . This choice allows to recover the standard Einstein equations in  $D$  dimensions with nonvanishing cosmological constant  $\Lambda_D$ , i.e.,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{D-2}{2}\Lambda_D g_{\mu\nu} = -\kappa T_{\mu\nu} \quad (1.15)$$

where the particular choice of the  $\Lambda_D$  normalization will be explained below. Let us note that the equations (1.13) again hold. Notice that in this case the new piece in the previous equation (1.15) proportional to  $\Lambda_D$  can be moved to the r.h.s. and then an energy-momentum tensor  $(T_{\Lambda_D})_{\mu\nu}$  can be defined as follows

$$(T_{\Lambda_D})_{\mu\nu} \equiv \frac{D-2}{2} \frac{\Lambda_D}{\kappa} g_{\mu\nu}. \quad (1.16)$$

In this case, both density and pressure from the cosmological constant contribution may be written for any number of dimensions as:

$$\rho_{\Lambda_D} \equiv \frac{D-2}{2} \frac{\Lambda_D}{\kappa} ; \quad P_{\Lambda_D} \equiv -\frac{D-2}{2} \frac{\Lambda_D}{\kappa} \quad (1.17)$$

since  $P_{\Lambda_D} = -\rho_{\Lambda_D}$  is the state equation for a cosmological constant.

Finally let us point out that the equations (1.10) may be expressed *à la Einstein* by writing all extra terms due to the  $f(R)$  presence on the r.h.s. One can try to recover the standard form of the Einstein equations as follows

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{-\kappa}{1+f_R} (T_{\mu\nu} + T_{\mu\nu}^{eff}) \quad (1.18)$$

where an effective energy-momentum tensor has been defined as

$$T_{\mu\nu}^{eff} \equiv \frac{1}{\kappa} \left[ \mathcal{D}_{\mu\nu} f_R - \frac{1}{2}(f(R) - R f_R) g_{\mu\nu} \right]. \quad (1.19)$$

This energy-momentum tensor does not necessarily obey the strong energy condition which holds in ordinary fluids (dust matter, radiation, etc.) do.

## 1.4 Equivalence with Brans-Dicke theories

From a classical field theory perspective, it is always possible to redefine the fields of a given theory in order to express the field equations in a more attractive way which would be easier either to handle or to solve. The price to pay is to introduce new auxiliary fields and even to perform either renormalizations or conformal transformations.

It is widely assumed that two theories are dynamically equivalent if, under a suitable redefinition of either gravitational or matter fields, one can make the field equations to coincide. Nevertheless, some controversy has appeared in recent times especially when conformal transformations are used to redefine fields (see for instance [28] and [29] and references therein).

As mentioned in Section 1.1 a possibility to construct alternative theories of gravity are the scalar-tensor theories which are based upon the introduction of an extra scalar field which modifies the gravitational sector. Those theories are still metric theories in the sense that the newly introduced fields do not couple to the fluid contributions.

The gravitational action for a general scalar-tensor theory in  $D$  dimensions is

$$S_{ST} = \int d^D x \sqrt{|g|} \left[ \frac{y(\phi)}{2} R - \frac{\omega(\phi)}{2} (\partial_\mu \phi \partial^\mu \phi) - U(\phi) \right]. \quad (1.20)$$

By choosing  $y(\phi) = \phi/\kappa$ ,  $\omega(\phi) = \omega_0/(\kappa\phi)$  and  $U(\phi) = V(\phi)/\kappa$ , the action

$$S_{BD} = \frac{1}{2\kappa} \int d^D x \sqrt{|g|} \left[ \phi R - \frac{\omega_0}{\phi} (\partial_\mu \phi \partial^\mu \phi) - V(\phi) \right] \quad (1.21)$$

is obtained from (1.20). This is the action for the Brans-Dicke theories which is obviously a particular case of scalar-tensor theories.

It can be shown that  $f(R)$  gravities within the metric formalism are nothing but a Brans-Dicke theory with Brans-Dicke parameter  $\omega_0 = 0$ . This fact is easily proven as follows: a new field  $\chi$  is introduced and for the sake of simplicity let us define

$$F(R) \equiv R + f(R). \quad (1.22)$$

Thus the action (1.3) can be seen to be equivalent to the action

$$S_\chi = \frac{1}{2\kappa} \int d^D x \sqrt{|g|} \left[ F(\chi) + \frac{dF(\chi)}{d\chi} (R - \chi) \right] \quad (1.23)$$

since if a variation of (1.23) with respect to  $\chi$  is performed, the equation which is found reads:

$$\frac{d^2 f(\chi)}{d\chi^2} (R - \chi) = 0 \quad (1.24)$$

and thus  $\chi = R$  provided  $d^2 f(\chi)/d\chi^2 \neq 0$ . Therefore the original action (1.3) is recovered. Defining now the scalar field  $\phi$  as  $\phi \equiv dF(\chi)/d\chi$  and introducing a potential  $V(\phi)$  as follows

$$V(\phi) \equiv \chi(\phi)\phi - F(\chi(\phi)) \quad (1.25)$$

the action (1.23) takes the form

$$S_\phi = \frac{1}{2\kappa} \int d^D x \sqrt{|g|} (\phi R - V(\phi)) \quad (1.26)$$

which is exactly the same as (1.21) if  $\omega_0 = 0$  is imposed.

By including the corresponding fluid sector given by an energy-momentum tensor  $T_{\mu\nu}$ , the field equations derived from (1.26) are

$$G_{\mu\nu} = -\frac{\kappa}{\phi} T_{\mu\nu} - \frac{1}{2\phi} g_{\mu\nu} V(\phi) + \frac{1}{\phi} \mathcal{D}_{\mu\nu} \phi \quad (1.27)$$

$$R = \frac{dV(\phi)}{d\phi} \quad (1.28)$$

where the trace of (1.27)

$$(D-1)\square\phi + \frac{D}{2}V(\phi) + \frac{2-D}{2}\phi \frac{dV}{d\phi} = -\kappa T \quad (1.29)$$

gives the dynamics of  $\phi$  in terms of the matter content.

Let us finally note that if  $f_{RR} \equiv d^2 f(R)/dR^2$  vanishes, the equivalence between the two theories cannot be guaranteed as can be seen from equation (1.24). On the other hand, the resulting Brans-Dicke equivalent theory makes clear that  $f(R)$  gravity theories have just one more extra degree of freedom than standard EH gravity. The apparent absence of kinetic term in the action (1.26) must not be thought of as the absence of dynamics in  $\phi$  since this scalar is dynamically related to the matter fields, as can be seen from expression (1.29). Thus  $\phi$ , or equivalently  $f(R)$ , is indeed a dynamical degree of freedom.

## 1.5 Geometrical results

In this section we present different geometrical results obtained from the modified Einstein equations which were obtained in Section 1.3. Particular interest will be devoted in Subsection 1.5.1 to vacuum solutions. Then, the possibility of mimicking the usual GR results using  $f(R)$  functions will be addressed in Subsection 1.5.2. These results were originally presented in [26].

### 1.5.1 Vacuum solutions

Let us consider the EH action (1.2) in  $D$  dimensions with nonvanishing cosmological constant. In this case the equations (1.15) can be studied in vacuum, i.e.  $T_{\mu\nu}$  vanishes for all its components and therefore

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \frac{D-2}{2}\Lambda_D g_{\mu\nu} = 0 \quad (1.30)$$

whose solutions satisfy

$$R_{\mu\nu} = \Lambda_D g_{\mu\nu} ; \quad R = D\Lambda_D \quad (1.31)$$

which motivated our choice for  $\Lambda_D$  normalization in Section 1.3. Equations (1.31) provide the conditions to be accomplished by a metric  $g_{\mu\nu}$  to allow vacuum solution in this case. If now one considers the  $f(R)$  general case provided by the equations (1.10), one may wonder about the condition for the existence of constant curvature solutions,  $R_0$  from now on, in a vacuum scenario. Thus, the equations (1.10) may be simplified to become

$$R_{\mu\nu} (1 + f_R) - \frac{1}{2} g_{\mu\nu} (R + f(R)) = 0. \quad (1.32)$$

Note that the term involving  $\mathcal{D}_{\mu\nu} f_R$  in (1.10) has disappeared since it vanishes when constant curvature is assumed. Taking the trace in the previous equation we get

$$2R(1 + f_R) - D(R + f(R)) = 0. \quad (1.33)$$

If  $R_0$  is a root of the previous equation, an effective cosmological constant may be defined as  $\Lambda_D^{eff} \equiv R_0/D$ . Provided the condition  $1 + f'(R_0) \neq 0$  is satisfied,  $R_0$  fulfills:

$$R_{\mu\nu} = \frac{R_0 + f(R_0)}{2(1 + f_R(R_0))} g_{\mu\nu}. \quad (1.34)$$

Let us illustrate this procedure considering a simple model:

$$f(R) = \frac{g_1}{R} + g_0 \quad (1.35)$$

which has been widely studied in the literature (see for instance [30] where  $D = 4$  and  $g_0 = 0$ ). Then the constant curvature solutions – for an arbitrary number of dimensions  $D$  – are

$$R_0 = \frac{-Dg_0 \pm \sqrt{D^2(g_0^2 - 4g_1) + 16g_1}}{2(D - 2)} \quad (1.36)$$

which reduce for  $D = 4$  to the expression

$$R_0 = -g_0 \pm \sqrt{g_0^2 - 3g_1}. \quad (1.37)$$

For the EH case in  $D = 4$  with cosmological constant  $\Lambda \equiv \Lambda_4$ , i.e.  $g_1 = 0$  and  $g_0 = -2\Lambda$ , the constant curvature solutions are both  $R_0 = 4\Lambda$  and  $R_0 = 0$  and for the vanishing cosmological constant case, i.e.  $g_0 = 0$ ,  $R_0 = \pm\sqrt{-3g_1}$  is obtained.

As a different approach, one can consider equation (1.33) as a differential equation for the  $f(R)$  function so that the corresponding solution would admit any curvature  $R$  value. The solution of (1.33) is just:

$$f(R) = \alpha R^{D/2} - R \quad (1.38)$$

where  $\alpha$  is an arbitrary constant. Thus the gravitational action (1.3) becomes

$$S_G = \frac{\alpha}{2\kappa} \int d^D x \sqrt{|g|} R^{D/2} \quad (1.39)$$

which has solutions of constant curvature for arbitrary  $R$ . The reason is that this action is scale invariant since the ratio  $\alpha/\kappa$  is a dimensionless constant.

### 1.5.2 Some EH solutions reproduced by $f(R)$ theories

Now we shall address the issue of finding some general criteria to mimic, by using general  $f(R)$  gravities, some solutions of the EH action not necessarily of constant scalar curvature and either with or without a cosmological constant term.

Let the metric tensor  $g_{\mu\nu}$  be a solution of EH gravity with cosmological constant, i.e. such that the equations (1.15) are fulfilled. Then the same metric tensor  $g_{\mu\nu}$  will be a solution for (1.10) provided the following compatibility equation

$$f_R R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} [f(R) + (D - 2)\Lambda_D] + \mathcal{D}_{\mu\nu} f_R = 0 \quad (1.40)$$

is fulfilled. Note that the fluid content comprised in  $T_{\mu\nu}$  has been considered to be strictly the same as in (1.15). This allowed us to cancel this term out in order to obtain the

compatibility equation (1.40). In Section 2.6, a slight deviation of fluid contents between EH and  $f(R)$  approaches will be permitted.

Some particularly interesting cases in which to apply this approach are the following:

1. The simplest case is obviously vacuum, i.e.  $T_{\mu\nu} \equiv 0$ , with vanishing cosmological constant  $\Lambda_D = 0$ . Then the equations (1.15) become:

$$R_{\mu\nu} = \frac{1}{2}Rg_{\mu\nu} \quad (1.41)$$

which imply  $R_0 = 0$  and  $R_{\mu\nu} = 0$ . Consequently  $g_{\mu\nu}$  is also a solution of any  $f(R)$  gravity provided the following condition

$$f(0) = 0 \quad (1.42)$$

is accomplished as seen from (1.40). This is for instance the case if  $f(R)$  is analytical around  $R = 0$  and it can be written as follows:

$$f(R) = \sum_{n=1}^{\infty} f_n R^n. \quad (1.43)$$

2. If the cosmological constant is different from zero ( $\Lambda_D \neq 0$ ), but still  $T_{\mu\nu} \equiv 0$ , the constant curvature results given in (1.31) are again obtained. Then the compatibility equation (1.40) reduces to (1.33) with  $R_0 = D\Lambda_D$ . In other words,  $g_{\mu\nu}$  is also a solution of the  $f(R)$  case provided

$$f(D\Lambda_D) = \Lambda_D(2 - D + 2f_R(D\Lambda_D)). \quad (1.44)$$

Notice also that in this situation, i.e. nonvanishing  $\Lambda_D$  and vacuum, according to the result in (1.39) there would also be a solution for any  $R_0$  in the particular case  $f(R) = \alpha R^{D/2} - R$ .

3. If the considered case is  $\Lambda_D = 0$  and conformal matter ( $T \equiv T^\mu_\mu = 0$ ), then the equations (1.15) would imply

$$R_0 = 0 ; R_{\mu\nu} = -\kappa T_{\mu\nu} \quad (1.45)$$

which will have a metric tensor  $g_{\mu\nu}$  as solution. Therefore, provided

$$f(0) = 0 ; f_R(0) = 0, \quad (1.46)$$

the same  $g_{\mu\nu}$  is also a solution of any  $f(R)$  gravity. This result could have particular interest in cosmological calculations for ultrarelativistic matter (i.e. conformal) dominated universes.

4. Again in the conformal matter case with nonvanishing  $\Lambda_D$ , constant curvature

$$R_0 = D\Lambda_D \quad (1.47)$$

is a solution for (1.15) for a given metric  $g_{\mu\nu}$  which is also a solution of  $f(R)$  provided again that the condition (1.44) is satisfied.

5. Finally for the general case with no assumption about  $\Lambda_D$  nor about  $T_{\mu\nu}$ , the metric tensor  $g_{\mu\nu}$  will be a solution for any  $f(R)$  gravity but for a modified energy momentum tensor  $\bar{T}_{\mu\nu}$  given by:

$$\bar{T}_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{\kappa} \left\{ f_R R_{\mu\nu} - \frac{1}{2} [f(R) + (D-2)\Lambda_D] g_{\mu\nu} + \mathcal{D}_{\mu\nu} f_R \right\}. \quad (1.48)$$

## 1.6 Constraints on $f(R)$ theories to ensure viability

$f(R)$  gravity models turn out to be severely constrained in order to provide consistent theories of gravity. In this section we review both cosmological and strictly gravitational conditions presented in [31]. Some relevant bibliography will also be provided.

The usual four conditions that are required for a viable  $f(R)$  theory are:

1.  $f_{RR} \geq 0$  for high curvatures [32]. This is the requirement for a classically stable high-curvature regime and for the existence of a matter dominated phase in the cosmological evolution. In the opposite case, an instability, referred to in the literature as the 'Dolgov-Kawasaki' or 'Ricci scalar' or 'matter' instability, would appear. Indeed, if  $f_{RR}$  is smaller than zero, then the extra degree of freedom of the theory would behave as a ghost. This stability condition may also be recovered in studies of cosmological perturbations [33] and it can be given a simple physical interpretation as in [34] where if an effective  $D$  dimensional gravitational constant is defined as  $G_{eff} \equiv G_D/(1 + f_R)$  then

$$\frac{dG_{eff}}{dR} = -\frac{f_{RR}}{(1 + f_R)^2} G_D. \quad (1.49)$$

It is easy to notice from the previous equation that if  $f_{RR} < 0$ ,  $G_{eff}$  would increase as  $R$  grows since  $R$  itself generates larger and larger curvature via equation (1.12). Such a mechanism would act to destabilize the theory with no stable ground state since if a small curvature starts growing it will do so without limit and the system would run away. If on the contrary  $f_{RR} \geq 0$ , a negative feedback mechanism operates to compensate the growth of  $R$  and consequently the runaway behaviour will not appear <sup>1</sup>.

---

<sup>1</sup>Note that in this analysis  $1 + f_R$  has been supposed to be positive (i.e.  $G_{eff} > 0$ ) as will be required from the second condition below to ensure viability.

2.  $1 + f_R > 0$  for all Ricci scalar curvature values. This condition ensures the effective Newton's constant to be positive at all times as can be seen from equation (1.18) and the graviton energy to be positive. This condition will also be proven in Chapter 4 to be required to recover standard thermodynamics of Schwarzschild-anti-de Sitter BHs in  $f(R)$  theories.

3.  $f_R < 0$  ensures ordinary GR behaviour is recovered at early times. Together with the condition  $f_{RR} > 0$ , it implies that  $f_R$  should be negative and a monotonically growing function of  $R$  in the range  $-1 < f_R < 0$ .

4.  $|f_R| \ll 1$  at recent epochs. This is imposed by local gravity tests [33], although it is still not clear what is the actual limit on this parameter and some controversy still remains about the required  $|f_R|$  value [20, 35]. This condition also implies that the cosmological evolution at late times resembles that of  $\Lambda$ CDM. In any case, this constraint is not required if one is only interested in building models for cosmic acceleration.

Let us summarize this section by saying that viable  $f(R)$  models can be constructed to be compatible with local gravity tests and other cosmological constraints [36].

## 1.7 Brane-world theories

As mentioned in the Motivation section, many of the SM extensions try to solve open issues in modern physics such as the *hierarchy problem*. Some approaches try to answer those questions by introducing extra spatial dimensions, where the number of dimensions of the total space (bulk space)  $D = 4 + \delta$ . Those attempts were first proposed independently by Kaluza [37] and Klein [38] and many proposals followed throughout the twentieth century [39, 40].

First proposals for large extra dimensions were provided in [40]: in this model, the SM matter is confined in a spatial 3-dimensional manifold and the brane itself is considered not to be a gravitational source. Hence the background metric is assumed to be Minkowskian. Gravitational fields are the only fields able to propagate through the whole bulk space. Therefore gravity also propagates in the extra  $\delta$  dimensions which are for simplicity often compactified in a toroidal shape whereby all extra dimensions acquire a radius  $R_B$ .

One of the most important consequences of this hypothesis is the relation between the fundamental gravitational scale in  $D$  dimensions  $M_D$  and the Planck scale  $M_P$  which is not a fundamental constant any more but the effective gravitational constant in the theory reduced to 4 dimensions. In fact one may write

$$M_P^2 \equiv V_\delta M_D^{2+\delta} \quad (1.50)$$

where  $V_\delta$  is the compactified volume in  $\delta$  dimensions [40], for instance in the toroidal

case  $V_\delta = (2\pi R_B)^\delta$ . The expression (1.50) allows to reduce the fundamental gravitational scale to the electroweak scale,  $M_D \sim \text{TeV}$ , if extra dimensions are large enough. For instance, compactification scales of  $R_B^{-1} \sim 10^{-3} \text{ eV}$  to  $10 \text{ MeV}$  provide this effect for extra dimensions  $\delta \sim 2$  and  $7$  respectively. By reducing the fundamental scale to  $M_D$ , gravitational effects may be detectable in experiments involving energies of this order [40] as will be explained in the Section 1.9.

## 1.8 Excitations in brane worlds: branons

Since no relativistic object may be considered as rigid in relativistic theories, the 3-brane, when embedded in the total space-time, may present fluctuations. These fluctuations were originally studied in [41]. In the extra dimensions of the BW models, such fluctuations are usually referred to as branons. They give rise to new states whose low energy dynamics has been widely studied [42, 43, 44]. A vast bibliography can be found dealing with branons [45, 46], their predicted detection in future colliders experiments [47] and the explanation that they may provide for the origin of dark matter [25, 48].

Let us consider a  $D$  dimensional bulk space  $\mathcal{M}_D$  wherein the brane lies embedded and that for simplicity we shall assume to be factorized in the form  $\mathcal{M}_D = \mathcal{M}_4 \times B$  where  $\mathcal{M}_4$  is a 4-dimensional space-time and  $B$  is a  $\delta$ -dimensional compactified manifold. The brane is therefore assumed to lie on the  $\mathcal{M}_4$  space-time manifold. As already mentioned, the gravitational contribution of the brane itself will not be considered.

Let us denote the coordinates over the manifold  $\mathcal{M}_D$  as  $\{x^\mu, y^m\}$  with  $\mu = 0, 1, 2, 3$  and  $m = 1, 2, \dots, \delta$  and the ansatz for the total space  $\mathcal{M}_D$  bulk metric will be

$$G_{MN} = \begin{pmatrix} \tilde{g}_{\mu\nu}(x) & \\ & -\tilde{g}'_{mn}(y) \end{pmatrix} \quad (1.51)$$

with signature  $(+, -, -, -; -, \dots, -)$ .

In the absence of the 3-brane, this metric possesses an isometry group that is assumed to be of the form  $G(\mathcal{M}_D) = G(\mathcal{M}_4) \times G(B)$ . The presence of the brane spontaneously breaks the symmetry to some subgroup  $G(\mathcal{M}_4) \times H$  with  $H \subset G(B)$  some subgroup of  $G(B)$ . Therefore the quotient space  $K = G(\mathcal{M}_D)/(G(\mathcal{M}_4) \times H) = G(B)/H$  may be defined.

The position of the brane can be parameterized as  $Y^M(x) \equiv \{x^\mu, Y^m(x)\}$  where the first four coordinates of the total space have been chosen to be the space-time coordinates corresponding to the brane  $\{x^\mu\}$ . Let us assume that the brane is located at a point on  $B$ , i.e.,  $Y_0 \equiv \{Y^m(x)\}$  corresponds to the fundamental state of the brane. In this case its induced metric in the ground state is just  $g_{\mu\nu} \equiv \tilde{g}_{\mu\nu} \equiv G_{\mu\nu}$ . However, when brane

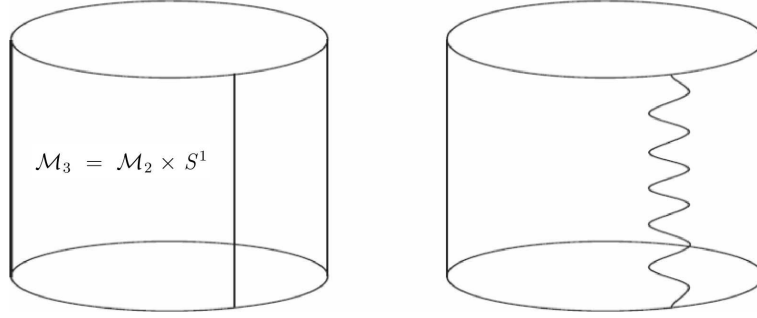


Figure 1.1: Brane with trivial topology in  $\mathcal{M}_3 = \mathcal{M}_2 \times S^1$  as originally presented in reference [49]. The fundamental brane state is plotted on the left whereas on the right side an excited state is presented.

excitations (branons) are present, the induced metric becomes

$$g_{\mu\nu} = \partial_\mu Y^M \partial_\nu Y^N G_{MN} = \tilde{g}_{\mu\nu} - \partial_\mu Y^m \partial_\nu Y^n \tilde{g}'_{mn}. \quad (1.52)$$

This situation may be illustrated by the simple Figure 1.1 where a 1-brane (string) is represented within a total space with two spatial coordinates  $\mathcal{M}_3 = \mathcal{M}_2 \times S^1$ .

Since the brane creation mechanism is in principle unknown, or at least out of the scope of the present section, let us assume that the brane dynamics is described by an effective action, so we are allowed to consider for this action the most general expression which is invariant under brane coordinates reparametrizations. Therefore, it is very common to perform an expansion in derivatives of the induced metric given by equation (1.52) to describe the brane dynamics. Then, the first order of this effective action would describe the brane dynamics at low energies and it is usually referred to as the Dirac-Nambu-Goto ( $NG$ ) action:

$$S_{NG} = -f^4 \int d^4x \sqrt{|g|} \quad (1.53)$$

where a constant  $f$  with energy units appears, which may be identified with the brane tension  $\tau \equiv f^4$  and  $d^4x \sqrt{|g|}$  is the brane volume element. As was mentioned above, the presence of the brane will break any existing isometry of  $B$  except those which leave the point  $Y_0$  on  $B$  invariant. In other words, the group  $G(B)$  is spontaneously broken to  $H(Y_0)$  denoting the  $Y_0$  isotropy group.

The brane excitations with respect to the broken Killing fields in  $B$  correspond to the zero modes and they are parameterized by the branon fields  $\pi^\alpha(x)$ ,  $\alpha = 1, \dots, k$  where  $k \equiv \dim(G(B)) - \dim(H)$ . These fields  $\pi^\alpha(x)$  may be interpreted like the corresponding coordinates in the quotient manifold  $K = G(B)/H$ .

In particular, for a fundamental state independent of the position  $Y_0$  of the brane in the  $B$  space, the action of an element of  $G(B)$  over  $Y_0$  will take  $Y_0$  to the point on  $B$  with coordinates

$$Y^m(x) \equiv Y^m(Y_0, \pi^\alpha(x)) = Y_0^m + \frac{1}{\sqrt{2\kappa}f^2} \xi_\alpha^m(Y_0) \pi^\alpha(x) + \mathcal{O}(\pi^2) \quad (1.54)$$

where the branons fields normalization is performed through  $\kappa = 8\pi/M_P^2$ . At this stage it is important to stress that coordinates for the transformed point given by (1.54) only depend on  $\pi^\alpha(x)$ , i.e., on the corresponding transformation parameters of the broken generators.

If  $B$  is considered to be an homogeneous space, the isotropy group does not depend on the particular chosen point where the brane lies, i.e.  $H(Y_0) = H$ . In this case  $B$  is homeomorphic to the coset  $K = G(B)/H$  which is the space of the Goldstone bosons associated to the spontaneous isometry breaking - transverse translations - produced by the presence of the brane. Thus the transverse translations of the brane - branons - can be considered as Goldstone bosons on the coset  $K$  and the branon fields can be defined as coordinates  $\pi^\alpha$  on  $K$ , which are chosen to be proportional to  $B$  coordinates, since the number of Goldstone bosons is equal to  $\dim(B)$ , as:

$$\pi^\alpha = \frac{v}{R_B} \delta_m^\alpha Y^m \quad (1.55)$$

where

$$v = f^2 R_B \quad (1.56)$$

is the typical size of the coset  $K$  and  $R_B$  is the typical size, in length units, of the compactified space  $B$ .

Therefore, according to the previous assumption (1.55), it is obvious that

$$\partial_\mu Y^m(x) = \frac{\partial Y^m}{\partial \pi^\alpha} \partial_\mu \pi^\alpha = \frac{1}{\sqrt{2\kappa}f^2} \xi_\alpha^m(Y_0) \partial_\mu \pi^\alpha + \mathcal{O}(\pi^2) \quad (1.57)$$

and the induced metric on the brane (1.52) is rewritten in terms of the branon fields  $\pi$  as

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} - \frac{1}{f^4} h_{\alpha\beta}(\pi) \partial_\mu \pi^\alpha \partial_\nu \pi^\beta \quad (1.58)$$

where  $h_{\alpha\beta}$  is the  $K$  metric which is easily obtained from the  $B$  metric

$$h_{\alpha\beta}(\pi) = f^4 \tilde{g}'_{mn}(Y(\pi)) \frac{\partial Y^m}{\partial \pi^\alpha} \frac{\partial Y^n}{\partial \pi^\beta} \quad (1.59)$$

as explained in [45]. In more complicated cases in which translational isometries in the bulk space are not only spontaneously but also explicitly broken, the metric  $g_{\mu\nu}$  could also

be a function of the extra dimension coordinates  $\{y^m\}$ . Then it is possible to show that branons may become massive. In fact in [25] these massive branons were shown to behave as WIMPs and thus form natural candidates for dark matter in this kind of scenario.

Therefore for small brane excitations in a background metric  $\tilde{g}_{\mu\nu}$ , the effective action (1.53) can be expressed as a derivative expansion as follows:

$$S_{eff}[\pi] = S_{eff}^{(0)}[\pi] + S_{eff}^{(2)}[\pi] + S_{eff}^{(4)}[\pi] + \dots \quad (1.60)$$

where the corresponding zeroth order is

$$S_{eff}^{(0)}[\pi] = -f^4 \int_{\mathcal{M}_4} d^4x \sqrt{|\tilde{g}|}. \quad (1.61)$$

Note that  $S_{eff}^{(2)}[\pi]$  and  $S_{eff}^{(4)}[\pi]$  hold for contributions to the effective action containing two and four derivatives of the branon fields respectively. Let us finish this digression by remarking that the term  $S_{eff}^{(2)}[\pi]$ , with two field derivatives, is nothing but the non-linear sigma model action associated to the coset space  $K$ .

## 1.9 Brane-skyrmions

Apart from branons, the brane may support other states due to the nontrivial homotopies of the coset space  $K$  such as strings, monopoles or skyrmions. This fact appears due to the possibility of wrapping around the extra dimension space  $B$  giving rise to nontrivial topological configurations as was studied in detail in the reference [49].

In fact, texture-like configurations, called brane-skyrmions, arise when the third homotopy group of  $K$  is nontrivial. In particular, for

$$\pi_3(B) = \pi_3(K) = \mathbb{Z}, \quad (1.62)$$

the third homotopy group will be the minimal one supporting the existence of those non-trivial configurations<sup>2</sup>. Those brane-skyrmions can be nicely understood in geometrical terms as some kind of holes [50] in the brane which make it possible to pass through them along the  $B$  space. This is because in the core of the topological defect the symmetry is reestablished. In particular, in the case we are interested in, the broken symmetry is basically the translational symmetry along the extra-dimensions.

In order to simplify the calculations, we shall consider an homogeneous compactified manifold  $B$  and the coset space  $K$  homeomorphic to  $SU(2)$  and equivalently to  $S^3$ . Then

$$B \simeq K \simeq SU(2) \simeq S^3. \quad (1.63)$$

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<sup>2</sup>Note that  $\pi_3(B) = \pi_3(K)$  if  $B$  is an homogeneous space.

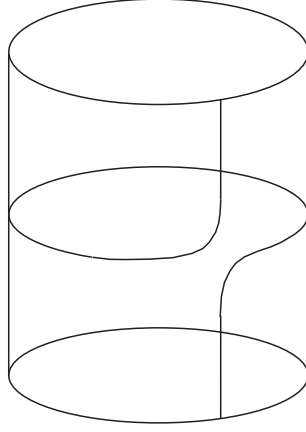


Figure 1.2: Brane-skyrmion configuration with  $n_W = 1$  and nonvanishing size in  $\mathcal{M}_3 = \mathcal{M}_2 \times S^1$  as originally presented in [49].

This fact allows to hold a third homotopy group  $\mathbb{Z}$ .

Furthermore, let us introduce spherical coordinates on both spaces,  $\mathcal{M}_4$  and  $K$  as follows: in  $\mathcal{M}_4$  we denote the coordinates  $\{t, r, \theta, \phi\}$  with  $\phi \in [0, 2\pi)$ ,  $\theta \in [0, \pi]$  and  $r \in [0, \infty)$ . On the coset manifold  $K$ , the spherical coordinates are denoted  $\{\chi_K, \theta_K, \phi_K\}$  with  $\phi_K \in [0, 2\pi)$ ,  $\theta_K \in [0, \pi]$  and  $\chi_K \in [0, \pi]$ . Notice that such coordinates cover the whole spherical manifolds and relate to the physical branon fields (local normal geodesic coordinates on  $K$ ) by:

$$\begin{aligned}\pi_1 &= v \sin \chi_K \sin \theta_K \cos \phi_K, \\ \pi_2 &= v \sin \chi_K \sin \theta_K \sin \phi_K, \\ \pi_3 &= v \sin \chi_K \cos \theta_K.\end{aligned}\tag{1.64}$$

The coset metric in spherical coordinates is written as

$$h_{\alpha\beta} = \begin{pmatrix} v^2 & & \\ & v^2 \sin^2(\chi_K) & \\ & & v^2 \sin^2(\chi_K) \sin^2(\theta_K) \end{pmatrix}.\tag{1.65}$$

### 1.9.1 Static brane-skyrmions

For static configurations, it can be proven that the mass for the brane-skyrmion may be obtained directly as:

$$M[\pi] = - \int_{\mathcal{M}_3} d^3x \mathcal{L}_{eff} = f^4 \int_{\mathcal{M}_3} d^3x \sqrt{|g|}\tag{1.66}$$

where the effective Lagrangian comes from the expression (1.53). In general this expression is divergent due to the contribution of the zeroth order term when  $\sqrt{|g|}$  is expanded in branon fields derivatives, reflecting the fact that the brane is an infinite object with finite tension. To prevent that, we subtract this term in order to get a brane-skyrmion finite mass, i.e.:

$$M_S[\pi] \equiv M[\pi] - M[0] = f^4 \int_{\mathcal{M}_3} dx^3 \sqrt{|g|} - M[0]. \quad (1.67)$$

As was shown in the previous section, the  $\pi^\alpha$  fields are mappings from the  $\mathcal{M}_4$  manifold to the coset manifold  $K$ . For static, i.e. time independent, field configurations, these could be understood as mappings from the corresponding spatial 3-dimensional hypersurface ( $\mathcal{M}_3$ ) to the coset space ( $S^3$  in the case we are studying). For finite energy configurations, fields should vanish at the spatial infinity and  $\mathcal{M}_3$  can be compactified to  $S^3$ . Therefore one may write  $\pi^\alpha : S^3 \rightarrow S^3$ . Since the third homotopy group of  $S^3$  is  $\mathbb{Z}$ , the mappings can be classified by an integer number  $n_W$ . Thus, branons can be identified with the topologically trivial configurations  $n_W = 0$ , whereas those configurations with  $n_W \neq 0$  will be denoted as brane-skyrmions.

Consequently, for static skyrmions this mapping may be implemented in the following way:

$$\phi_K = \phi ; \quad \theta_K = \theta \quad \chi_K = F(r) \quad (1.68)$$

with the boundary conditions  $F(0) - F(\infty) = n_W \pi$  for a winding number  $n_W \neq 0$ .

In this case,  $M_S[\pi]$  may be written as a  $F(r)$  functional and the correct mass for this kind of skyrmions is obtained by minimizing  $M_S[F]$  in the space function with adequate boundary conditions.

From expression (1.67), it can be proven that the skyrmion is point-like, stable and its mass becomes:

$$M_S = 2\pi^2 f^4 R_B^3 \quad (1.69)$$

## 1.10 Gravitational signatures at the LHC

As already mentioned, it can be seen from the equation (1.50), the fundamental gravitational scale could be as low as the electroweak scale if extra dimensions are large enough. Since the LHC is operating at a center of mass energy of  $\sqrt{s} = 14 \text{ TeV}$ , if the fundamental scale of gravitation is  $M_D \sim \text{TeV}$ , both the production and decay of Schwarzschild mini BHs at high ratio becomes possible [51].

These BHs once produced would decay into SM particles with a clean signature and a low background. Several features of such objects could then be extracted from experimental data: for instance BH masses  $M_{\text{BH}}$  may be determined very precisely due to the absence of missing energy and their temperature could be extracted from the energy spectrum of the products. Thus the correlation between these two quantities may provide relevant information able to determine the number of extra dimensions, and therefore the fundamental scale of gravity. On the other hand the Hawking evaporation law could be tested experimentally.

The total cross section when two partons collide at the LHC with an impact parameter less than the Schwarzschild radius  $R_S$  is of order

$$\sigma(M_{\text{BH}}) \approx \pi R_S^2 = \frac{1}{M_D^2} \left[ \frac{M_{\text{BH}}}{M_D} \left( \frac{8\Gamma(\frac{D-1}{2})}{D-2} \right) \right]^{\frac{2}{D-3}} \quad (1.70)$$

and it does not contain small coupling constants. If  $M_D \sim \text{TeV}$  the cross section is of order  $\text{TeV}^{-2} \approx 400 \text{ pb}$  and therefore BHs will be produced copiously. The total production cross section ranges from 0.5 nb for  $M_D = 2 \text{ TeV}$ ,  $D = 11$  to 120 fb for  $M_D = 6 \text{ TeV}$   $D = 7$ . For  $M_D \sim 1 \text{ TeV}$ , the LHC – with a peak of luminosity of  $30 \text{ fb}^{-1}/\text{year}$  – will produce  $10^7$  BH/year.

Experimental signatures rely on two qualitative properties: on the one hand, the absence of small couplings as seen from expression (1.70) and on the other hand, the flavor independence nature of BHs decays as will be explained in the following paragraph. Note that when  $M_{\text{BH}}$  approaches  $M_D$ , some stringy corrections to the previous assumptions may arise but semiclassical arguments remain valid as long as  $M_{\text{BH}} \gg M_D$ .

Once the BHs have been produced they decay following a process governed by their Hawking temperature  $T_H \sim 1/R_S$  with an associated wavelength  $\lambda = 2\pi/T_H$  larger than the BH size and therefore BHs would emit, in a first approximation, as point radiators mostly in the s-waves. This indicates that BHs decay equally to particles on the brane and in the bulk since the decay is only sensitive to the radial coordinate. If the approximation  $M_{\text{BH}} \gg T_H$  is made, the average multiplicity of particles  $\langle N \rangle$  produced in the BH

evaporation is given by:

$$\langle N \rangle = \frac{2\sqrt{\pi}}{D-3} \left( \frac{M_{\text{BH}}}{M_D} \right)^{\frac{D-2}{D-3}} \left( \frac{8\Gamma\left(\frac{D-1}{2}\right)}{D-2} \right)^{\frac{1}{D-3}}. \quad (1.71)$$

Since the decay is thermal, it does not discriminate between particle species (of the same mass and spin) and therefore BHs decay, roughly speaking, with the same probability to all SM particles. The signal of hard primary leptons and hard photons is quite clean with a negligible background since the production of SM leptons or photons occur at much smaller rate than BH production [51].

The way to determine  $M_{\text{BH}}$  and  $T_H$  deals with the study of decay products and the fits of the energy spectrum of those products to the Planck formula respectively. Once those two quantities are determined, they could provide some evidence of the Hawking radiation and of the fact that the observed events indeed come from BH evaporation and not from any other mechanism.

The relation between those two quantities,  $M_{\text{BH}}$  and  $T_H$  obtained independently, may shed light about the dimensionality of the space since it can be proved that

$$\log(T_H) = -\frac{1}{D-3} \log(M_{\text{BH}}) + \text{constant} \quad (1.72)$$

where the constant does not depend on  $M_{\text{BH}}$ . Therefore the previous equation provides a direct method to determine the dimensionality  $D$  of the space as the slope of this relation.

The experimental signatures outlined above allow us to state that if the fundamental scale of gravitation is of order TeV, as suggested in BW scenarios, some important physical consequences may appear. In fact, colliders study of BHs – eventually produced at a high rate in accelerators such as LHC – could help revealing the main features of physics in the vicinity of the electroweak scale or even determining the total number of dimensions of the space-time.

# Chapter 2

## Dark energy in $f(R)$ theories

### 2.1 Introduction

As was commented in Chapter 1, when the modified Einstein equations were rewritten *à la Einstein*, the presence of a function  $f(R)$  in the gravitational sector modifying the usual EH Lagrangian may be understood as the introduction of an effective fluid which is not restricted to hold the usual energy conditions. Therefore  $f(R)$  functions may be used to explain the present cosmological acceleration. Historically, some  $f(R)$  models were proposed to modify GR at short scales, i.e., high energies trying to explain inflation, as for instance  $f(R) \propto R^2$ , but no interest was paid in those models to provide a mechanism to cause late time acceleration. First attempts to induce cosmological acceleration considered  $f(R) \propto 1/R$  but those models turned out to be in conflict with solar system tests [52] and even to be unstable when matter is introduced [53].

Before studying these issues, let us mention that  $f(R)$  models, apart from satisfying those gravitational and cosmological conditions given in Section 1.6, should verify some extra conditions of cosmological viability. For instance, they have to include a background evolution with Big Bang nucleosynthesis (BBN) and both radiation and matter dominated cosmological eras. This fact will be explicitly studied in this chapter in Section 2.6. On the other hand, they must provide cosmological perturbations compatible with cosmological constraints from CMB and large scale structures (LSS). This fact will be studied thoroughly in Chapter 3.

The present chapter is organized as follows: in Section 2.2 we shall revise the standard approach to describe the cosmological evolution in the  $\Lambda$ CDM model in a homogeneous, isotropic and spatially flat metric. In the following Section 2.3 we shall generalize the usual Einstein equations when  $f(R)$  gravity theories are present. Then we shall study in Section 2.4 the cosmological viability conditions for  $f(R)$  theories to hold a dust matter

dominated era followed by a late time acceleration and some interesting models which have been proposed to be viable will be provided in Section 2.5. Then, Section 2.6 will be the core of the chapter and it will be devoted to study if  $f(R)$  theories are able to mimic standard  $\Lambda$ CDM evolution. These  $f(R)$  models will possess vacuum solutions with null scalar curvature what allows to recover some GR solutions usually considered.

To finish this chapter, we shall study in Section 2.7 how the modification of the gravitational sector by  $f(R)$  models may mimic the influence of perfect fluids (parameterized by a constant equation of state) in the cosmological evolution without any presence of such a fluid in the fluid content. The chapter will finish with Section 2.8 by drawing some attention over the main obtained conclusions.

The results presented in this chapter were originally published in [54].

## 2.2 Standard Einstein equations in a FLRW universe

Since the *leitmotiv* of this chapter is to study cosmological solutions, our universe, which is assumed to be isotropic and homogeneous at large enough scales for fundamental observers, may be represented with a  $D = 4$  dimensional Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = dt^2 - a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega_2^2 \right) \quad (2.1)$$

expressed in cosmic time  $t$  and where  $a(t)$  is usually referred to as the scale factor. Alternatively, this metric may be expressed in conformal time  $\tau$ , defined by the relation  $dt \equiv a(\tau)d\tau$  and thus this metric becomes

$$ds^2 = a^2(\tau) \left( d\tau^2 - \frac{dr^2}{1 - kr^2} - r^2 d\Omega_2^2 \right) \quad (2.2)$$

In this metric, the Hubble parameter may be defined in either cosmic or conformal time as

$$H(t) \equiv \frac{da(t)/dt}{a(t)} \equiv \frac{\dot{a}}{a} ; \quad \mathcal{H} \equiv \frac{da(\tau)/d\tau}{a(\tau)} \equiv \frac{a'(\tau)}{a(\tau)} \quad (2.3)$$

respectively and the identity  $aH \equiv \mathcal{H}$  is straightforwardly inferred.

For the values of the parameter  $k$  smaller, equal or bigger than zero, the universe is spatially hyperbolic, flat or spherical respectively. In the following calculations we shall be considering  $k = 0$ . This choice is justified according to WMAP data [11] where the results obtained for  $\Lambda$ CDM model are:  $\Omega_k \equiv -k/H_0^2$ , with  $H_0 \equiv H(t_{today}) = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$

and  $h = 0.699 \pm 0.018$  and  $-0.0133 < \Omega_k < 0.0084$  (95% CL). Therefore terms related with  $k$  will be subdominant in either Friedmann's or generalized Friedmann equations to be presented in following the section.

Considering the previously introduced metric (2.1) and perfect fluids given by (1.7) for the present fluids, the only two independent Einstein equations for  $D = 4$  are the Friedmann and the acceleration equations respectively, which may be written in cosmic time as:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \sum_{\alpha} \rho_{\alpha} \quad (2.4)$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{6} \sum_{\alpha} (\rho_{\alpha} + 3P_{\alpha}) \quad (2.5)$$

where  $G \equiv G_4$  is the gravitational constant in four dimensions and the summation over subindex  $\alpha$  holds for the present fluids contributions (baryons, radiation, dark matter, DE, etc.). If positive cosmological acceleration is required, i.e.  $\ddot{a} > 0$ , the condition to be accomplished from expression (2.5) would be  $\sum_{\alpha} (\rho_{\alpha} + 3P_{\alpha}) < 0$ . This condition would require that if only a perfect fluid is present, its state equation would satisfy the condition  $\omega_{\alpha} < -1/3$  which is not the case for standard fluids, such as for instance dust matter and radiation. On the contrary a cosmological constant does provide positive acceleration in equation (2.5) since its state equation satisfies  $\omega_{\Lambda} = -1$ .

For this metric, the energy-momentum conservation equations lead, in cosmic and conformal time respectively, to the following equations:

$$\begin{aligned} \dot{\rho}_{\alpha} + 3(1 + \omega_{\alpha})H\rho_{\alpha} &= 0 \\ \rho'_{\alpha} + 3(1 + \omega_{\alpha})\mathcal{H}\rho_{\alpha} &= 0 \end{aligned} \quad (2.6)$$

which hold separately for each fluid whose state equations are assumed to be  $P_{\alpha} = \omega_{\alpha}\rho_{\alpha}$ . Previous equation is integrated to give:

$$\rho_{\alpha}(t) = \rho_{\alpha}(t_0) \left(\frac{a(t_0)}{a(t)}\right)^{3(1+\omega_{\alpha})} \quad (2.7)$$

where  $t_0$  is an arbitrary time and the corresponding scale factor is  $a(t_0)$ .

By using the definition given in (1.5) and (1.6), the Ricci scalar curvature for FLRW spatially flat metric is written in terms of the scale factor and its derivatives as follows

$$R = 6 \left[ \left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a} \right] = \frac{6}{a^2} (\mathcal{H}' + \mathcal{H}^2). \quad (2.8)$$

Let us finish this section by rewriting the previous Friedmann equation (2.4). To do so, let us divide that equation by  $H_0^2 \equiv H^2(t_0)$  and consider as present fluids dust matter ( $\omega_M = 0$ ), radiation ( $\omega_{\text{Rad}} = 1/3$ ) and cosmological constant  $\Lambda$  ( $\omega_\Lambda = -1$ ). Thus, taking into account (2.7) for each present fluid, we get:

$$\frac{H^2(t)}{H_0^2} = \Omega_M a(t)^{-3} + \Omega_{\text{Rad}} a(t)^{-4} + \Omega_\Lambda \quad (2.9)$$

where we have used the notation:

$$\Omega_M \equiv \frac{8\pi G \rho_M(t_0)}{3H_0^2(t_0)} ; \quad \Omega_{\text{Rad}} \equiv \frac{8\pi G \rho_{\text{Rad}}(t_0)}{3H_0^2(t_0)} ; \quad \Omega_\Lambda \equiv \frac{\Lambda}{3H_0^2(t_0)} \quad (2.10)$$

and the normalization of the scale factor  $a(t_0) = 1$  has been used. Note that if expression (2.9) is evaluated at  $t = t_0$ , then  $\Omega_M + \Omega_{\text{Rad}} + \Omega_\Lambda \equiv 1$ .

## 2.3 Modified Einstein equations in a FLRW universe

Inserting the metric (2.1) for  $D = 4$  in the equations (1.10) and assuming also energy-momentum tensor as given in (1.7) for a fluid with energy density  $\rho_0$  and pressure  $P_0$ , the only independent modified Einstein equations are

$$3(1 + f_R)\frac{\ddot{a}}{a} - \frac{1}{2}(R + f(R)) - 3\frac{\dot{a}}{a}\dot{R}f_{RR} = -8\pi G \rho_0 \quad (2.11)$$

$$(1 + f_R)(\dot{H} + 3H^2) - \frac{1}{2}(R + f(R)) - \frac{1}{a}\frac{d}{dt}\left(a^2\dot{R}f_{RR}\right) = 8\pi G P_0 \quad (2.12)$$

and in conformal time  $\tau$ , these equations are given by

$$\frac{3\mathcal{H}'}{a^2}(1 + f_R) - \frac{1}{2}(R + f(R)) - \frac{3\mathcal{H}}{a^2}f'_R = -8\pi G \rho_0 \quad (2.13)$$

$$\frac{1}{a^2}(\mathcal{H}' + 2\mathcal{H}^2)(1 + f_R) - \frac{1}{2}(R + f(R)) - \frac{1}{a^2}(\mathcal{H}f'_R + f''_R) = 8\pi G P_0. \quad (2.14)$$

Remind that dot denotes here derivative with respect to time  $t$  whereas  $\tau$  derivative was denoted with prime. A very useful equation to use in the following calculations is the combination (2.14) minus (2.13) which becomes

$$2(1 + f_R)(-\mathcal{H}' + \mathcal{H}^2) + 2\mathcal{H}f'_R - f''_R = 8\pi G(\rho_0 + P_0)a^2. \quad (2.15)$$

At this stage we should note that, for instance, according to equations (2.11), (2.12) together with (2.8), it is clear that modified Einstein equations are not second order in derivatives any more, but at least third order in the scale factor derivatives provided  $f_{RR} \neq 0$ .

## 2.4 Cosmological viability for $f(R)$ dark energy models

In this section we shall revise the conditions that a model for DE given by a  $f(R)$  theory must fulfill in order to be cosmologically viable: i.e., any viable  $f(R)$  model should have a matter dominated phase long enough to provide the adequate cosmological evolution prior to a late time acceleration phase. As a matter of fact, equations (2.11) and (2.12) with dust matter as the unique present fluid, can be rewritten in the form of a system of autonomous equations [55]. In that reference two variables,  $m$  and  $r$  are introduced as follows:

$$m \equiv \frac{R f_{RR}}{1 + f_R} ; \quad r \equiv -\frac{R(1 + f_R)}{R + f(R)}. \quad (2.16)$$

Both the dynamics and stability of that autonomous system are determined by six critical points  $P_{1,\dots,6}$  – according to the notation in [55] – that appear in the system resolution.

### 2.4.1 Critical points and stability

According to the results presented in [55], both points  $P_5$  and  $P_6$  satisfy

$$m(r) = -r - 1. \quad (2.17)$$

If in the previous equation  $m$  is assumed to be constant, the condition (2.17) holds straightforwardly from two other equations in the autonomous system. In this case the points  $P_{2,\dots,6}$  always exist while  $P_1$  and  $P_4$  are present for values  $m = 1$  and  $m = -1$  respectively. The critical points  $P_5$  and  $P_6$  which give the exact matter era evolution, i.e.,  $a(t) \propto t^{2/3}$ , exist only for  $m = 0$  ( $P_5$ ) or for  $m = -(5 \pm \sqrt{73})/12$  ( $P_6$ ) but the latter corresponds to a vanishing matter density and obviously it does not give a standard matter era.

If on the contrary  $m$  is not assumed to be constant, the number of solutions depends on the particular  $f(R)$  choice, but only  $P_{1,5,6}$  can be accelerated and only  $P_5$  might give rise to matter era. This last situation would require  $m \simeq 0$  to resemble the standard matter era evolution. Summarizing the result in this case, only trajectories passing near  $P_5$  with  $m \simeq 0$  at  $r \simeq -1$  and landing on an accelerated attractor would give a viable cosmological evolution.

### 2.4.2 Classification of $f(R)$ models

By studying all possible trajectories in the  $m$  and  $r$  variables of the already mentioned autonomous system, it can be shown [55] that a classification of  $f(R)$  models can be

based entirely upon geometrical properties of the curve  $m(r)$ . These two variables allow to classify the  $f(R)$  models in four different classes: **I**, **II**, **III** and **IV** depending on the existence of a standard matter epoch and a final accelerated expansion as follows: on the one hand, the viable matter dominated epoch requires

$$m \approx +0 ; \quad \frac{dm}{dr} > -1 \quad (2.18)$$

at  $r \approx -1$ . On the other hand, the late time acceleration epoch requires to fulfill one of the following two conditions: a de Sitter acceleration follows the matter epoch if and only if

$$1. \quad 0 \leq m(r) \leq 1 \text{ at } r = -2 \quad (2.19)$$

whereas a non-phantom accelerated attractor follows the matter dominated epoch if and only if

$$2. \quad m = -r - 1 ; \quad \frac{\sqrt{3} - 1}{2} < m \leq 1 ; \quad \frac{dm}{dr} < -1. \quad (2.20)$$

For instance, according to the previous requirements over  $m$  and  $r$  variables, models of the type  $f(R) = \alpha R^{-n} - R$  and  $f(R) = \alpha R^{-n}$  do not satisfy these conditions for any  $n > 0$  and  $n < -1$  and are consequently cosmologically nonviable.

The main features of each class of models are:

**Class I:** this class covers all the cases for which the curve  $m(r)$  does not connect the accelerated attractor with the standard matter point  $(r, m) = (-1, 0)$  either because  $m(r)$  does not pass near that matter point, i.e.,  $m(r \rightarrow -1) \neq 0$ , or because the branch of  $m(r)$  that accelerates is not connected with the standard matter point. Moreover, instead of having a standard matter phase given by a scale factor  $a(t) \propto t^{2/3}$ , these  $f(R)$  models possess a peculiar scale factor behaviour  $a(t) \propto t^{1/2}$  before accelerating epoch and are therefore unsuitable models.

**Class II:** for these  $f(R)$  models the  $m(r)$  curve does connect the upper vicinity ( $m > 0$ ) of  $(r, m) = (-1, 0)$  with a critical point able to provide acceleration. Therefore models here have a matter epoch and are asymptotically equivalent (hardly distinguishable) to  $\Lambda$ CDM model ( $\omega_{eff} \equiv -1 - 2\dot{H}/3H^2 = -1$ ), i.e., they are asymptotically de Sitter and observationally acceptable. These models satisfy both equations (2.18) and (2.19).

**Class III:** these  $f(R)$  models may possess an approximated matter era but as a transient state followed by a final and strongly phantom attractor at late-time. This

$f(R)$ models	$m(r)$	Class I	Class II	Class III
$-R + \alpha R^{-n}$	$-1 - n$	$n > -0.713$	—	$-1 < n < -0.713$
$\alpha R^{-n}$	$-\frac{n(1+r)}{r}$	$n > 0$	$n \in (-1, 0), \alpha < 0$	—
$-R + R^p [\log(\alpha R)]^q$	$\frac{(p+r)^2}{qr} - 1 - r$	$p \neq 1$	$p = 1, q > 0$	—
$-R + R^p \exp qR$	$-r + \frac{p}{r}$	$p \neq 1$	—	—
$-R + R^p \exp(q/R)$	$-\frac{p+r(2+r)}{r}$	$p \neq 1$	$p = 1$	—

Table 2.1: Classification of some  $f(R)$  DE models presented in [55]. None of these models belongs to **Class IV**. Models that belong to **Class II** for the provided parameter intervals, at least satisfy the conditions to have a matter era followed by a de Sitter attractor.

is due to the fact that the  $m(r)$  curve intersects the critical line  $m(r) = -r - 1$  at  $-1/2 < m < 0$ . The approximated matter era is a very fast transient phase and only a narrow range of initial conditions may allow it. Since matter era is practically unstable, these models are generally ruled out by the observations.

**Class IV:** for models of this class the connection between the upper vicinity of the point  $(r, m) = (-1, 0)$  to the region located on the critical line  $m(r) = -r - 1$  is possible. Therefore these models are observationally acceptable: they possess an approximate standard matter epoch followed by a non-phantom acceleration with an effective equation of state  $\omega_{eff} \equiv -1 - 2\dot{H}/3H^2 > -1$ , thus these models possess a standard DE behaviour. These models satisfy both equations (2.18) and (2.20).

**Classes II and IV** have therefore some chance to be cosmologically viable but the basin of the attractor has to be determined to provide acceptable trajectories according to the already mentioned analysis fully performed in [55]. In Table 2.1 the previous analysis have been applied to some  $f(R)$  models usually considered in the literature.

## 2.5 Some cosmologically viable $f(R)$ models

In this section we provide three  $f(R)$  models already presented in the literature which claim to be cosmologically viable.

$$a) f(R) = \lambda R_0 \left[ \left( 1 + \frac{R^2}{R_0^2} \right)^{-n} - 1 \right]$$

This model was originally considered in reference [56] with  $n, \lambda > 0$  and  $R_0$  of the order of the presently observed effective cosmological constant. Then  $f(0) = 0$  and the cosmological constant is claimed to disappear in flat space-time but  $f_{RR}(0)$  is negative and therefore, according to condition 1 in Section 1.6, flat space-time would be unstable.

For scalar curvatures  $R \gg R_0$ ,  $f(R)$  tends to  $\lambda R_0$  and the model would behave as the EH case with an effective cosmological constant. On the other hand, de Sitter space-time with curvature  $R_1 > 0$  is also a vacuum solution provided  $R_1(f_R(R_1) - 1) = 2f(R_1)$  according to equation (1.33) for  $D = 4$  and  $R = R_1$ . Thus, this case would present an effective cosmological constant  $\Lambda(R_1) = R_1/4$ .

For this model it can be proved that conditions **1** and **2** of Section 1.6 are satisfied in the curvature interval  $[R_1, \infty)$  if they are accomplished at  $R = R_1$ . Then these two conditions hold if and only if

$$\left[1 + \left(\frac{R_1}{R_0}\right)^2\right]^{n+2} > 1 + (n+2)\left(\frac{R_1}{R_0}\right)^2 + (n+1)(2n+1)\left(\frac{R_1}{R_0}\right)^4. \quad (2.21)$$

Note that condition **3** in that section is straightforwardly satisfied if as considered, parameters  $\lambda$  and  $n$  are positive. On the other hand, this model also satisfies the required conditions to provide a matter dominated era at  $R \gg R_0$  and does not possess the already mentioned Dolgov-Kawasaki instability. The remaining condition **4** is easily accomplished if  $R_0$  is considered much smaller than  $R$  at recent epochs.

A simple choice of parameters  $\lambda$ ,  $n$  and  $R_0$  shows that this model obeys the conditions (2.18) and (2.19) and therefore according to the analysis in the previous section it possess a matter dominated epoch and a de Sitter late time acceleration.

$$b) f(R) = -\alpha m_1 \left(\frac{R}{\alpha}\right)^n \left[1 + \beta \left(\frac{R}{\alpha}\right)^n\right]^{-1}$$

This model was first proposed in reference [33] in order to mimic  $\Lambda$ CDM evolution in the high-redshift regime and to accelerate at low redshift with an expansion history close to  $\Lambda$ CDM model. In this model, the parameter  $n$  was considered to be positive and for convenience the mass scale  $\alpha$  was given by

$$\alpha \equiv \frac{8\pi G \bar{\rho}_0}{3} = (8315 \text{ Mpc})^{-2} \left(\frac{\Omega_M h^2}{0.13}\right) \quad (2.22)$$

where  $\bar{\rho}_0$  is the average density today. This model does not have a bare cosmological constant since  $f(0) = 0$  and its parameters  $m_1$ ,  $\beta$  and  $n$  may be rewritten as [57]

$$\frac{m_1}{\beta} \approx 6 \frac{1 - \Omega_M}{\Omega_M} \quad (2.23)$$

$$\frac{m_1}{\beta^2} = -\frac{f_R(R_0)}{n} \left(\frac{12}{\Omega_M} - 9\right)^{n+1} \quad (2.24)$$

where  $\Omega_M$  is the effective matter energy density at the present time. Finally the constraint  $|f_R(R_0)| < 0.1$  was imposed and  $R_0$  is the scalar curvature today as would be obtained

from  $\Lambda$ CDM model, i.e.

$$R_0 \approx \alpha m_1 \left( \frac{12}{\Omega_M} - 9 \right). \quad (2.25)$$

Setting the values  $n = 1$  and  $\Omega_M = 0.3$ , it was proven in reference [57] that this model belongs to the Class **II** presented in the previous section.

$$c) \ f(R) = -\alpha R_* \log \left( 1 + \frac{R}{R_*} \right)$$

This two-parameter  $f(R)$  model presented in [58], where parameters  $\alpha$  and  $R_*$  are positive, has claimed to be cosmologically viable and different from  $\Lambda$ CDM. In fact, it does satisfy both cosmological conditions (2.18) and (2.19) presented in previous Section 2.4, provided  $\alpha > 1$  and regardless of the value of  $R_*$ .

Conditions **1**, **2** and **3** given in Section 1.6 are also satisfied but concerning the condition **4** also given in that section, it will be shown in Section 3.4 that this  $f(R)$  theory does not satisfy this condition, showing that this model is indeed distinct from  $\Lambda$ CDM. This fact and its consequences will be studied in Section 3.4.

## 2.6 $f(R)$ with no cosmological constant

Now that the modified Einstein equations have been presented for FLRW metric, some interesting results at cosmological scales will be obtained in this section. The presented approach tries to mimic cosmological well-known GR results in different cosmological eras employing adequate  $f(R)$  functions. Reconstruction procedures of this kind have been widely studied in the literature [59, 60, 61] where by rewriting the involved equations in new variables and assuming a given cosmological solution, mainly in vacuum, the required  $f(R)$  gravity is obtained.

In this section the addressed issue will be to find a  $f(R)$  gravity able to reproduce the current cosmic speed-up appearing in standard  $\Lambda$ CDM cosmology. This function is required to be analytical at  $R = 0$  and to have  $R = 0$  as a vacuum solution, therefore it will not contain any cosmological constant contribution. From a more formal point of view we are seeking for some  $f(R)$  gravity model having the same FLRW solution as the standard EH action with cosmological constant for nonrelativistic matter (dust matter, i.e.  $P_M = 0$ ). For the searched cosmological constant absence it is clear that the  $f(R)$  expansion at  $R = 0$  must start at the  $R^2$  term to avoid, on the one hand, having cosmological constant and, on the other hand, redefining the gravitational constant.

### 2.6.1 Cosmological evolution in $\Lambda$ CDM model

Let us solve Einstein's equations (1.15) for the standard EH action plus a cosmological constant  $\Lambda$  with dust matter in the energy-momentum tensor side.

The most recent cosmological data quoted in reference [11] are compatible at late times with a cosmological model based on a spatially flat FLRW metric like (2.1) together with Einstein's equations with a cosmological constant  $\Lambda \neq 0$  and dust matter (including dark matter). In this case, the equations (1.15) will be valid and matter content will be written in terms of a pressureless perfect fluid

$$T^\mu_\nu = \text{diag}(\rho_{M0}(t), 0, 0, 0). \quad (2.26)$$

Equation  $\mu = \nu = 0$  (time-time component) in (1.15) becomes

$$\left(\frac{\dot{a}_0(t)}{a_0(t)}\right)^2 = \frac{8\pi G}{3}\rho_{M0}(t) + \frac{\Lambda}{3} \quad (2.27)$$

where  $\rho_{M0}(t)$  in previous equation is given by expression (2.7) if  $\omega_{\alpha=M}$  is fixed to 0. Thus

$$\rho_{M0}(t) = \rho_{M0}(t_0) \left(\frac{a_0(t_0)}{a_0(t)}\right)^3 \quad (2.28)$$

where the 0 subindex means that the standard EH equations (1.15) with a cosmological constant are being considered. This notation will be relevant later on when standard  $\Lambda$ CDM cosmology will be compared with the results coming from the action (1.3) for the  $f(R)$  function that we shall find in this section.

Substituting the expression (2.28) in the equation (2.27), the solution  $a_0(t)$ , using the notation introduced in Section 2.2 in this chapter, is found to be:

$$a_0(t) = \left(\frac{\Omega_M}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3} \left(\frac{3\sqrt{\Omega_\Lambda}}{2} H_0 t\right). \quad (2.29)$$

On the other hand, by taking the trace of (1.15) in this case, i.e. cosmological constant and dust matter, it is found that

$$R_0(t) - 4\Lambda = 8\pi G \rho_{M0}(t). \quad (2.30)$$

### 2.6.2 $f(R)$ case with no cosmological constant

Now let us consider the equations (1.10) but in the case where no cosmological constant is considered and the energy-momentum tensor for dust matter will be

$$T^\mu_\nu = \text{diag}(\rho_M(t), 0, 0, 0). \quad (2.31)$$

Then the equation (2.11) becomes

$$3(1 + f_R)\frac{\ddot{a}}{a} - \frac{1}{2}(R + f(R)) - 3\frac{\dot{a}}{a}\dot{R}f_{RR} = -8\pi G\rho_M \quad (2.32)$$

where we have eliminated the subindex 0 in the different quantities to avoid any confusion with the previous case presented in Subsection 2.6.1. As was already mentioned, it is clear that the solutions for equation (2.32) will strongly depend on the function  $f(R)$ : different choices for this function will lead to different evolutions of the universe for the same initial conditions. However, our approach to the problem will be to find a function  $f(R)$  so that the solution  $a(t)$  of the equation (2.32) will be exactly the same as the solution provided by the expression (2.29) that we obtained by using GR with nonvanishing cosmological constant and which seems to fit the present cosmological data. In other words, we want to find the  $f(R)$  model such that the solution for the equation (2.32) is exactly the scale factor (2.29), i.e.:

$$a(t) \equiv a_0(t) \quad (2.33)$$

for the same initial (or present, i.e.  $t = t_0$ ) conditions. If it were possible to find such a function  $f(R)$  then, it would be possible to avoid the necessity for introducing any cosmological constant just by considering a gravitational action such that given in the expression (1.3). In the following it will be shown that such a function happens to exist and its precise form will be provided. In order to do that one first notices that accomplishing the condition (2.33) after radiation-matter equality clearly implies

$$R = R(t) \equiv R_0(t) \quad (2.34)$$

and then  $R(t)$  and  $R_0(t)$  may be used indistinctly. On the other hand we shall write the matter density as the former matter density provided by expression (2.7) plus a new contribution, i.e.

$$\rho_M(t) = \rho_{M0}(t) + \Delta\rho(t), \quad (2.35)$$

accounting for a slight variation with respect to the density provided in the Subsection 2.6.1. Assuming that matter for arbitrary  $f(R)$  is still nonrelativistic in this cosmological era we have

$$\Delta\rho(t) = \Delta\rho(t_0) \left( \frac{a_0(t_0)}{a_0(t)} \right)^3 \quad (2.36)$$

where according to the expression (2.7) particularized for the constraint (2.33) and considering the relation given in (2.30) we can write

$$\left( \frac{a_0(t_0)}{a_0(t)} \right)^3 = \frac{R(t) - 4\Lambda}{8\pi G \rho_{M0}(t_0)} \quad (2.37)$$

and then (2.36) becomes

$$\Delta\rho(t) = -\eta \frac{R(t) - 4\Lambda}{\kappa} \quad (2.38)$$

where we have introduced the parameter

$$\eta \equiv -\frac{\Delta\rho(t_0)}{\rho_{M0}(t_0)} \quad (2.39)$$

so that matter density (2.35) is rewritten as

$$\rho_M(t; \eta) = (1 - \eta)\rho_{M0}(t) \quad (2.40)$$

Finally the last term on the l.h.s. of equation (2.32) can be written in terms of the scalar curvature by differentiating expression (2.30) with respect to cosmic time  $t$  and using the conservation equation (2.6). Hence, we get

$$\begin{aligned} 3(R - 3\Lambda)(R - 4\Lambda)f_{RR} + \left(-\frac{1}{2}R + 3\Lambda\right)f_R - \frac{1}{2}f(R) - \Lambda \\ - \eta(R - 4\Lambda) = 0 \end{aligned} \quad (2.41)$$

where the time dependence of  $R$  is implicit. This last equation can be considered as a second order linear differential equation for the function  $f(R)$ , so two initial conditions are needed to solve it: the natural choice that has been judged more convenient and physically meaningful is the following:

1. Firstly, the absence of any cosmological constant in the gravitational action is required, so that  $f(0) = 0$ .
2. Secondly, the standard EH action behaviour should be recovered for low scalar curvatures without redefining the Newton constant, i.e.  $f_R(0) = 0$ .

Moreover,  $f(R)$  function is wanted to be analytical at the origin so that  $R = 0$  should be a solution for the field equations in vacuum. This is an extremely important requirement since it allows both Minkowski and Schwarzschild solutions to be vacuum solutions.

With these initial conditions, the equation (2.41) can be solved by using standard methods. One particular solution is:

$$f_p(R) = -\eta R + 2(\eta - 1)\Lambda. \quad (2.42)$$

The homogeneous equation associated with (2.41) is a Gauss-type equation solved in terms of hypergeometric functions. The general solution of the homogeneous equation can be written as:

$$f_h(R) = \Lambda(K_+ f_+(R) + K_- f_-(R)) \quad (2.43)$$

where

$$f_{\pm}(R) = \left(3 - \frac{R}{\Lambda}\right)^{-a_{\pm}} {}_2F_1 \left[ a_{\pm}, 1 + a_{\pm} - c, 1 + a_{\pm} - a_{\mp}; -\left(3 - \frac{R}{\Lambda}\right)^{-1} \right] \quad (2.44)$$

and the symbol  ${}_2F_1$  holds for hypergeometric functions and the constants

$$a_{\pm} = -\frac{1}{12} (7 \pm \sqrt{73}) \quad ; \quad c = -1/2 \quad (2.45)$$

have been introduced. The  $\eta$ -dependent constants  $K_+$  and  $K_-$  must be determined from the initial conditions given above. Numerically it is found that:

$$K_+ = 0.6436 (-0.9058 \eta + 0.0596) \quad ; \quad K_- = 0.6436 (-0.2423 \eta + 3.4465). \quad (2.46)$$

The hypergeometric functions given in (2.44) are generally defined in the whole complex plane. However, a real gravitational action is wanted. In principle this requirement is very easy to achieve since the coefficients in the equation (2.41) and the constants  $K_{\pm}$  are all real. Then it is obvious that the real part of the functions appearing in the expression (2.44) is a proper solution of the homogeneous equation associated with (2.41). Thus the function we are seeking can be written as:

$$f(R) \equiv f_p(R) + \text{Re}[f_h(R)]. \quad (2.47)$$

Nevertheless, the situation is more complicated. The homogeneous equation has three regular singular points at  $R_1 = 3\Lambda$ ,  $R_2 = 4\Lambda$  and  $R_3 = \infty$ . This results in the solution  $f_h(R)$  having two branch points  $R_1$  and  $R_2$ . More concretely there are two cuts along the real axis: one from minus infinity to  $R_1$  and another from  $R_2$  to infinity. Thus one must be quite careful when interpreting (2.47). From minus infinity to  $R_1$  there is only one Riemann sheet of  $f_h(R)$  where  $f(0)$  and  $f_R(0)$  vanish and therefore this is the one that we have to use to define  $f(R)$ . From  $R_1$  to  $R_2$  the real part of  $f_h(R)$  is well defined. Finally from  $R_2$  to infinity there is only one Riemann sheet producing a smooth behaviour of  $f(R)$ . To reach this sheet one must understand  $R$  in the above equation as  $R + i\epsilon$ . At the present moment we do not know if this analytical structure has any fundamental meaning or it is just an artefact of our construction. Much more important is the fact that the function  $R + f_p(R) + f_h(R)$ , which is the analytical extension of our Lagrangian, is analytical at  $R = 0$ , having at this point the local behaviour  $R + \mathcal{O}(R^2)$ . Therefore our generalized gravitational Lagrangian  $R + f(R)$  does guarantee that  $R = 0$  is a vacuum solution as can be seen from expression (1.12). At the same time, this Lagrangian reproduces the current evolution of the universe without any cosmological constant.

Now that the  $f(R)$  function in (2.47) has been obtained, it is possible to check our result out by solving (2.11) in terms of  $a(t)$  for the  $f(R)$  given in (2.47). This is done by rewriting the equation (2.32) in terms of  $a(t)$  by using (2.8) and (2.7) together with the dust matter density in terms of  $\eta$  and  $a(t)$  as provided by expression (2.35). At this

stage, the consistency of our results has been checked out by introducing the scale factor (2.29) and  $f(R)$  given in (2.47) in (2.11) and both sides of the equation turned to be equal for all  $\eta$  values. Thus it has been guaranteed that our gravitational action proportional to  $R + f(R)$  provides the same cosmic evolution –in the required cosmological eras – as the EH action with cosmological constant  $\Lambda$  in a dust matter universe. Therefore, our model does verify, in the same range of precision, all the experimental tests that the standard cosmological model does in the present era. Notice also that in principle this can be achieved for any value of  $\eta$ , i.e. for any desired amount of matter. Nevertheless, some restrictions should be imposed on the parameter  $\eta$ . For instance it is obvious that in a dust matter dominated universe  $\rho_M(t; \eta) \geq 0$  implies  $\eta \leq 1$ .

Much more stringent bounds can be set on parameter  $\eta$  by demanding that this model works properly back in time up to Big Bang Nucleosynthesis (BBN) era. Observations indicate that the cosmological standard model fits correctly primordial light elements abundances during BBN, that means that the expansion rate  $H(t)$  cannot deviate from that of standard cosmology  $H_0(t)$  in more than 10% for the background evolution (see for instance [62] for further details). Therefore by the time of BBN, departure of our model from the standard cosmology must not be too large and the equation (2.11) should provide a similar behaviour to the one given by the standard Friedmann equation (2.27) where now the density will include both dust and radiation contributions.

At BBN the DE contribution is negligible compared with dust and radiation densities. The scalar curvature is of order  $10^{-39} \text{ eV}^2$  (with  $\hbar = c = 1$  for these calculations) and by that time dust and radiation densities are of the order of  $10^{16} \text{ eV}^4$  and  $10^{21} \text{ eV}^4$  respectively. Since  $R \simeq R_0$  we can rewrite the equation (2.11) as a modified Friedmann equation as follows

$$H^2(t) = H_0^2(t) \left\{ \frac{10^5 R - \eta R + \frac{1}{2}(Rf_R - f(R))}{10^5 R [1 + f_R - 3f_{RR}(1 - \eta)R]} \right\}. \quad (2.48)$$

As was commented above, to reproduce light elements abundances it is required that  $H^2(t) = H_0^2(t)(1 \pm 0.2)$  for curvatures of order  $R_{\text{BBN}}$ . This implies that the second factor on the r.h.s. of the previous expression (2.48) should be between 0.8 and 1.2 by that period. Thus in order to match our  $f(R)$  gravity model with the standard cosmology at the BBN times we need to tune  $\eta$  to a value about 0.065 with a stringent fine tuning. Therefore the matter content of our model is not too different from the one in the standard cosmology and the difference is in fact smaller than experimental precision in [11].

Concerning the problem of viability for this particular  $f(R)$  model, if conditions **2**, **3** and **4** given in Section 1.6 are required to hold, the parameter  $\eta$  has to be fixed to a fine tuned value  $\eta \approx -1.4311$  but for this value  $f_{RR}$  reverse its sign at high enough curvatures and therefore the condition **1** in that section is not accomplished. Therefore it may be stated that the  $f(R)$  model given by expression (2.47) should be considered as an

effective model able to reproduce  $\Lambda$ CDM model cosmological expansion after radiation-matter equality but not as a consistent gravitational theory valid for all scales.

## 2.7 Effective fluid description of $f(R)$ gravities

In this section we shall be interested in finding those  $f(R)$  functions such that the corresponding modified Einstein equations in vacuum exactly reproduce the cosmological evolution of EH gravity with a given perfect fluid, i.e., the introduced modifications of the EH action through a function  $f(R)$  will play the role of the fluid source.

Let us thus consider such a perfect fluid 'f' obeying the following barotropic state equation

$$P_f = \omega_f \rho_f \quad (2.49)$$

with constant  $\omega_f$  and whose density  $\rho_f$  scales according to the conservation equation (2.7) with the scale factor  $a \equiv a(t)$  as

$$\rho_f(a) = \rho_f(a(t_0)) \left( \frac{1}{a(t)} \right)^{3(1+\omega_f)} = \rho_f(a(t_0)) x^{1+\omega_f} \quad (2.50)$$

where  $\rho_f(a(t_0))$  is the value of the fluid density for a given value of the scale factor  $a(t_0) \equiv 1$ . For the sake of simplicity a new variable  $x$  has been introduced in the previous expression (2.50) defined as follows

$$x \equiv \frac{1}{a^3}. \quad (2.51)$$

With this new variable  $x$ , if the only present (or at least the dominant one) fluid is the defined above, the standard EH Friedmann and acceleration equations in cosmic time  $t$  are respectively

$$\begin{aligned} H^2 &= \frac{8\pi G \rho_f}{3} = H_0^2 x^{1+\omega_f} \\ \frac{\ddot{a}}{a} &= -\frac{8\pi G}{6} \rho_f (1 + 3\omega_f) = -\frac{1}{2} H_0^2 (1 + 3\omega_f) x^{1+\omega_f} \end{aligned} \quad (2.52)$$

where as usual  $8\pi G \rho_f(a(t_0)) \equiv 3H_0^2$ .

Analogously to the procedure in the previous section of this chapter, we shall consider the modified Einstein equations where there will not be any fluid contribution. The solution for these equations is wanted to be the same scale factor as the one which is the solution of equations (2.52). In other words, the presence of the fluid in the EH equations wants to be replaced by the contribution of some  $f(R)$  function in the modified Einstein equations.

To do so, note that the scalar curvature according its expression (2.8) for spatially flat FLRW metric may be rewritten as

$$R \equiv 6 \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a} \right] = 8\pi G(1 - 3\omega_f)\rho_f \quad (2.53)$$

and

$$\dot{R} \equiv \frac{dR}{dt} = -3H 8\pi G(1 + \omega_f)(1 - 3\omega_f)\rho_f(a(t_0)). \quad (2.54)$$

At this stage, let us introduce a dimensionless variable  $\tilde{R} \equiv R/H_0^2$ . Hence with this notation in variables  $\tilde{R}$  and  $x$  we get from expression (2.53) that

$$\tilde{R} = 3(1 - 3\omega_f) \left( \frac{1}{a} \right)^{3(1+\omega_f)} = 3(1 - 3\omega_f)x^{1+\omega_f}. \quad (2.55)$$

Note that for the specific choices  $\omega_f = -1, 1/3$  no relation between variables  $\tilde{R}$  and  $x$  may be straightforwardly established through (2.55). For those cases  $x$  has to be determined by solving the equations (2.52).

Since  $\dot{x} = -3Hx$  the following equalities can be written down

$$\frac{H^2}{H_0^2} = x^{1+\omega_f} \quad (2.56)$$

$$\dot{\tilde{R}} = -9H(1 - 3\omega_f)(1 + \omega_f)x^{1+\omega_f} \quad (2.57)$$

$$\frac{d\tilde{R}}{dx} \equiv \tilde{R}_x = 3(1 - 3\omega_f)(1 + \omega_f)x^{\omega_f} \quad (2.58)$$

$$\frac{d}{dx} \left( \frac{1}{\tilde{R}_x} \right) = -\frac{1}{\tilde{R}_x^2} 3(1 - 3\omega_f)(1 + \omega_f)\omega_f x^{\omega_f-1} \quad (2.59)$$

$$\frac{d\tilde{f}}{d\tilde{R}} = \frac{d\tilde{f}(x)}{dx} \frac{1}{\tilde{R}_x} ; \quad H_0^2 \frac{d^2 f}{dR^2} = \frac{d^2 \tilde{f}(x)}{dx^2} \frac{1}{\tilde{R}_x} - \frac{d\tilde{f}(x)}{dx} \frac{3}{\tilde{R}_x^3} (1 - 3\omega_f)(1 + \omega_f)\omega_f x^{\omega_f-1} \quad (2.60)$$

$$\frac{\dot{a}}{a} \dot{R} = -3H^2 x \tilde{R}_x H_0^2 \quad (2.61)$$

where  $\tilde{f}(\tilde{R}) \equiv f(R)/H_0^2$ .

Therefore, by considering a fluid density given by expression (2.50), the modified Friedmann equation given by (2.11) may be rewritten as a second order differential equation for  $f(R)$ . Analogously, this equation may be expressed as a differential equation in  $x$  variable for  $f(R(x)) \equiv f(x)$  and it becomes

$$\begin{aligned} \frac{d\tilde{f}(x)}{dx} \frac{1}{\tilde{R}_x} \left\{ 3 \frac{\ddot{a}}{a} \tilde{R}_x - 9H^2 [3(1 - 3\omega_f)(1 + \omega_f)\omega_f x^{\omega_f}] \right\} - \frac{1}{2} H_0^2 \tilde{f}(x) + 9H^2 \frac{x}{\tilde{R}_x} \frac{d^2 \tilde{f}(x)}{dx^2} \\ = 3H_0^2 x^{1+\omega_f} \end{aligned} \quad (2.62)$$

where expressions (2.52) and (2.58) have to be substituted in the previous equation.

The general solution of (2.62) is then:

$$\tilde{f}(x) = -3(1 - 3\omega_f)x^{1+\omega_f} + c_+ x^{\omega_f^+} + c_- x^{\omega_f^-} \quad (2.63)$$

where

$$\omega_f^\pm = \frac{1}{12} \left[ 9\omega_f + 7 \pm (9\omega_f^2 + 78\omega_f + 73)^{1/2} \right]. \quad (2.64)$$

Requiring  $\omega_f \neq -1, 1/3$  to avoid possible indeterminacies, it is possible to rewrite (2.63) in terms of  $\tilde{R}$  as

$$\tilde{f}(\tilde{R}) = -\tilde{R} + c_+ \left[ \frac{\tilde{R}}{3(1 - 3\omega_f)} \right]^{\frac{\omega_f^+}{1+\omega_f}} + c_- \left[ \frac{\tilde{R}}{3(1 - 3\omega_f)} \right]^{\frac{\omega_f^-}{1+\omega_f}}. \quad (2.65)$$

### 2.7.1 Some examples

Some interesting cases for the fluid content are the dust matter, radiation and cosmological constant fluids, i.e.,  $\omega_{M, \text{Rad}, \Lambda} = 0, 1/3, -1$  and  $\eta_f = \eta_{M, \text{Rad}, \Lambda}$  respectively. For these three cases, the corresponding functions become

$$\begin{aligned} \tilde{f}_M(x) &= c_+ x^{\frac{1}{12}(7+\sqrt{73})} + c_- x^{\frac{1}{12}(7-\sqrt{73})} - 3x \\ \tilde{f}_\Lambda(x) &= -12 + c_- x^{-1/3} + c_+ \\ \tilde{f}_{\text{Rad}}(x) &= c_+ x^{5/3} + c_- \end{aligned} \quad (2.66)$$

where constants  $c_\pm$  are arbitrary integration constants that can be fixed if either boundary or initial conditions are imposed.

## 2.8 Conclusions

In this chapter we have found the  $f(R)$  gravity which exactly reproduces the same evolution of the universe, from BBN up to the present time, as standard  $\Lambda$ CDM model does, but without the introduction of any form of DE or cosmological constant. The gravitational Lagrangian  $R + f(R)$  is analytical at the origin and consequently  $R = 0$  is a vacuum solution for the field equations. Therefore Minkowski, Schwarzschild and other important  $R = 0$  GR solutions, with  $\Lambda = 0$ , are also solutions for this  $f(R)$  gravity. This result was originally presented in [54].

The price that we have to pay for all those good properties is that our Lagrangian, considered as a function of  $R$ , has a very complicated analytical structure with cuts along the real axis from infinity to  $R = 3\Lambda$  and from  $R = 4\Lambda$  to infinity. Obviously the only reasonable interpretation of our action is as some kind of effective action. In classical physics one typically starts from some action principle, obtains the corresponding field equations and finally solves them for some initial or boundary conditions. In this work we have proceeded in the opposite way: we started from solutions obtained in the standard cosmological model and then we have searched for an action that, possessing certain properties, gives rise to field equations having the same solutions.

Classical actions are of course real but effective quantum actions usually have a complex structure coming from loops and related to unitarity. The presence of an imaginary part in the action, evaluated on some classical configuration, indicates quantum lost of stability by particle emission of this configuration [63]. Therefore it is tempting to think that our action could have some interpretation in terms of an effective quantum action. However, our action determination procedure does not allow to make such a kind of statement. Complications in the action could be just an artifact of our construction. In any case it was shown that such action exists and it reproduces the present universe evolution without DE having  $R = 0$  as a vacuum solution. As a drawback of this result, we have shown that this  $f(R)$  function cannot be considered as a fully consistent gravitational theory since it does not obey the viability conditions revised in Chapter 1 and that therefore it should be regarded as an effective model to mimic  $\Lambda$ CDM cosmological background evolution.

To conclude the chapter a completely general procedure to reproduce EH gravity with an arbitrary perfect fluid by using  $f(R)$  theories has been implemented. It has been explicitly shown that any perfect fluid when is described by a constant equation of state can be mimicked by an appropriate  $f(R)$  model. Standard cases for perfect fluids such as dust matter, radiation and cosmological constant have been presented in this analysis.

# Chapter 3

## Cosmological perturbations in $f(R)$ theories

### 3.1 Introduction

This chapter will be devoted to a study of the evolution of scalar cosmological perturbations in  $f(R)$  theories. To do so, a completely general procedure will be implemented and several consequences will be analyzed. The importance for addressing the present problem lies in the necessity to discriminate among different DE models, including  $f(R)$  modified gravities, by using observations. It is well-known that by choosing adequate  $f(R)$  functions, one can mimic any expansion history, and in particular that of the  $\Lambda$ CDM model. Accordingly, the exclusive use of observations from SNIa [64], baryon acoustic oscillations [65] or CMB shift factor [11], based upon different distance measurements which are sensitive only to the cosmological expansion history, cannot settle the question of the DE nature [66].

However, there exists a different type of observations which are sensitive, not only to the cosmological expansion history, but also to the evolution of matter density perturbations. Pioneering work on density perturbations in FLRW cosmological models was presented in [67]. The fact that the evolution of perturbations depends on the specific gravity model, i.e., it differs in general from that of Einstein's gravity even though the background evolution is the same, means that this kind of observations will help distinguishing between different models capable to explain cosmic acceleration. This is therefore the aim of the present chapter: to provide a completely exact method to determine how cosmological perturbations grow in  $f(R)$  theories and to settle the extracted consequences from this result. Such a problem has been exhaustively considered in the literature [68]. In this chapter we shall show that for  $f(R)$  theories the differential equation for the matter density

contrast is a fourth order differential equation in the longitudinal (also called conformal or Newtonian) gauge but it reduces to a second order differential equation for sub-Hubble modes.

This general equation will be compared with the standard simplification procedure (so-called quasi-static approximation) widely used in the previous literature. This approximation considers simplifications in the density equation determination from the very beginning. We shall show that for general  $f(R)$  functions, the quasi-static approximation is not justified. However, for those  $f(R)$  adequately describing the present phase of accelerated expansion and satisfying local gravity tests, it does give a correct description for the evolution of perturbations.

Once the general results are presented, some immediate applications may be implemented. For instance, our analysis may also be used to settle the validity of some proposed  $f(R)$  models, by comparing the predicted matter spectra with recent observations of LSS [69].

The present chapter is organized as follows: in Section 3.2 we briefly review the theory of cosmological perturbations for the standard  $\Lambda$ CDM model, introducing the gauge-invariant variables and revising the well-known results for the EH theory in order to establish a comparison with  $f(R)$  gravities. Next, Section 3.3 will be devoted to thoroughly study cosmological perturbations in  $f(R)$  theories. The perturbed modified Einstein equations are obtained through a completely general procedure for those theories in the Subsection 3.3.1. In Subsection 3.3.2 the density perturbations equation is obtained whereas the Subsection 3.3.3 compares the obtained exact results with the ones given by using the quasi-static approximation. Finally in this section, in Subsection 3.3.4 we shall study the growth of perturbations for some particular  $f(R)$  models. Section 3.4 will then show how our previous results may be used to constrain or rule some  $f(R)$  models out and finally some general conclusions for the presented results will be given in Section 3.5.

This chapter is based upon the results presented in references [70, 71, 72].

## 3.2 Theory of cosmological perturbations

### 3.2.1 Generalities

To study cosmological perturbations, the 4-dimensional full line element may be decoupled into background and perturbed parts as follows

$$ds^2 = g_{(0)\mu\nu}dx^\mu dx^\nu + \delta g_{\mu\nu}dx^\mu dx^\nu \quad (3.1)$$

$\mu, \nu = 0, 1, 2, 3$  with  $g_{(0)\mu\nu}$  representing the homogeneous FLRW background metric and  $\delta g_{\mu\nu}$  describing a small perturbation. This perturbation may be split [73] in three different types: scalar (S), vector (V) and tensor (T) contributions as follows

$$\delta g_{\mu\nu} = \delta g_{\mu\nu}^S + \delta g_{\mu\nu}^V + \delta g_{\mu\nu}^T. \quad (3.2)$$

This classification obviously refers to the way that fields included in  $\delta g_{\mu\nu}$  transform under three-space coordinate transformations on a constant time hypersurface. Tensor perturbations produce gravitational waves which do not couple to energy density and pressure inhomogeneities and propagate freely. Vector perturbations are dumped with cosmological expansion and are therefore negligible today. On the contrary, scalar perturbations may lead to growing inhomogeneities which will give rise to the large scale structures and the CMB anisotropies which are seen today. Thus let us explicitly implement each type of perturbations:

- Scalar perturbations: the most general metric perturbations of this type are given by four scalar functions:  $\phi$ ,  $\psi$ ,  $E$  and  $B$  of the space-time coordinates, as follows

$$\delta g_{\mu\nu}^S = a^2(\tau) \begin{pmatrix} 2\phi & -B_{,i} \\ -B_{,i} & 2(\psi\delta_{ij} - E_{,ij}) \end{pmatrix} \quad (3.3)$$

where  $i, j = 1, 2, 3$  from now on hold for spatial indices and subindex  $,i$  means ordinary derivative with respect to  $i^{th}$ -coordinate.

- Vector perturbations: these perturbations can be represented by two divergenceless three-vectors  $F_i$  and  $S_i$  as follows:

$$\delta g_{\mu\nu}^V = -a^2(\tau) \begin{pmatrix} 0 & -S_i \\ -S_i & F_{i,j} + F_{j,i} \end{pmatrix} \quad (3.4)$$

where divergenceless conditions - Einstein's convention applied - mean

$$F^i_{,i} = S^i_{,i} = 0 \quad (3.5)$$

and shift from upper to lower indices -and viceversa- is performed through the spatial part of spatially flat background metric tensor, i.e.,  $\delta_{ij}$  and its inverse  $\delta^{ij}$ .

- Tensor perturbations: tensor perturbations are given by a symmetric three-tensor  $h_{ij}$  satisfying the following conditions

$$h^i_i = 0 \ ; \ h^{ij}_{,j} = 0 \quad (3.6)$$

i.e., traceless and transversality conditions respectively, meaning that  $h_{ij}$  does not contain any piece transforming as scalars nor as vectors. Thus, the metric contribution  $\delta g_{\mu\nu}^T$  is

simply given by

$$\delta g_{\mu\nu}^T = -a^2(\tau) \begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix}. \quad (3.7)$$

The number of independent functions introduced to define  $\delta g_{\mu\nu}$  without loss of generality is ten: four scalar functions for scalar perturbations, two three-vectors for vector perturbations with one constraint each and one symmetric three-tensor with four conditions for tensor perturbations. This number coincides with the number of independent components of  $\delta g_{\mu\nu}$  as a  $4 \times 4$  symmetric tensor.

### 3.2.2 Gauge-invariant variables and gauge choice

Metric perturbations, as the ones defined above, are gauge-dependent, i.e. an infinitesimal coordinates transformation could give rise to two apparently different perturbations whereas they indeed represent the same physical perturbation. This is the reason why Bardeen introduced [74] gauge-invariant quantities that are explicitly invariant under infinitesimal coordinate transformations. The starting point is to consider infinitesimal coordinate transformations

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \xi^\mu(x). \quad (3.8)$$

It can be proven very easily that in the new coordinates  $\{\tilde{x}^\mu\}$  the metric  $g_{\mu\nu}(x)$  can be written as:

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \mathcal{L}_\xi g_{\mu\nu}(x) + \mathcal{O}(\xi^2) \quad (3.9)$$

what proves that two metrics  $g_{\mu\nu}$  and  $\tilde{g}_{\mu\nu}$  differing on a Lie derivative represent the same physical perturbation<sup>1</sup>. Let us consider a coordinate transformation given by the parameters  $(\xi^0, \xi^i)$ , i.e.,

$$\begin{aligned} \tilde{\tau} &= \tau + \xi^0 \\ \tilde{x}^i &= x^i + \xi^i = x^i + \bar{\xi}^i + \delta^{ij} \xi_{,j} \end{aligned} \quad (3.10)$$

where prime holds for derivative with respect to  $\tau$ , and  $\xi^i$  is decomposed as  $\xi^i = \bar{\xi}^i + \xi_{,j} \delta^{ji}$ , i.e., it is given by a solenoidal part,  $\bar{\xi}^i$ , and an irrotational part  $\xi_{,j} \delta^{ji}$  according to Helmholtz's theorem. Therefore  $d\tau$ ,  $dx^i$  and  $a(\tau)$  can be expressed in terms of the new

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<sup>1</sup>Lie derivative of a twice covariant tensor  $g_{\mu\nu}$  with respect to  $\xi$  is given by  $\mathcal{L}_\xi g_{\mu\nu} \equiv -g_\mu^\lambda \xi_{\lambda;\nu} - g_\nu^\lambda \xi_{\lambda;\mu} + g_{\mu\nu;\lambda} \xi^\lambda$  where  $g_{\mu\nu;\lambda} = 0$  if Levi-Civita connection is considered.

coordinates  $\{\tilde{x}^\mu\}$  as:

$$\begin{aligned}
d\tau &= d\tilde{\tau} - \xi^{0'} d\tilde{\tau} - \xi^0_{,i} d\tilde{x}^i \\
dx^i &= d\tilde{x}^i - \xi^{i'} d\tilde{\tau} - \xi^i_{,j} d\tilde{x}^j \\
&= d\tilde{x}^i - \left( \bar{\xi}^{i'} + \delta^{ij} \xi^{i'}_{,j} \right) d\tilde{\tau} - (\bar{\xi}^i_{,j} + \delta^{ik} \xi_{,kj}) d\tilde{x}^j \\
a(\tau) &= a(\tilde{\tau}) - \xi^0 a'(\tilde{\tau}).
\end{aligned} \tag{3.11}$$

Therefore if identities (3.11) are applied to the expression (3.1), the obtained metric should have the aspect of the original line element provided the involved quantities defined in expressions (3.3), (3.4) and (3.7) transform as follows:

- Scalar perturbations:

$$\begin{aligned}
\tilde{\Phi} &= \phi - \mathcal{H}\xi^0 - \xi^{0'} \\
\tilde{\Psi} &= \psi + \mathcal{H}\xi^0 \\
\tilde{B} &= B + \xi^0 - \xi' \\
\tilde{E} &= E - \xi
\end{aligned} \tag{3.12}$$

where only scalar contributions  $\xi^0$  and  $\xi$  are present.

- Vector perturbations:

$$\begin{aligned}
\tilde{F}_i &= F_i - \bar{\xi}^i \\
\tilde{S}_i &= S_i + \bar{\xi}^{i'}
\end{aligned} \tag{3.13}$$

where only vector contribution  $\bar{\xi}_i$  is present.

- Tensor perturbations:

$$\tilde{h}_{ij} = h_{ij} \tag{3.14}$$

which turn out to be gauge-invariant.

From previous results, gauge-invariant quantities can be constructed. For scalar perturbations, since four scalar functions were introduced and two scalar gauge parameters ( $\xi^0$  and  $\xi$ ) are present, two gauge-invariant quantities could be, for instance:

$$\Phi = \phi + \frac{1}{a}[(B - E')a]' \quad ; \quad \Psi = \psi + \mathcal{H}(B - E') \tag{3.15}$$

where by construction  $\Phi = \tilde{\Phi}$  and  $\Psi = \tilde{\Psi}$  are known as the Bardeen's potentials in [74].

For vector perturbations one gauge-invariant quantity could be

$$\mathcal{S}_i = S_i + F'_i. \quad (3.16)$$

With all previous results in mind, one can choose, i.e. one may specify in which coordinate system the scalar perturbations are going to be studied. There exist several possibilities for the gauge choice. Among them we can mention synchronous and longitudinal (or conformal-Newtonian) gauges.

*Synchronous gauge*: This gauge is defined by the conditions  $\phi = B = 0$  [67]. However, it can be shown that the required synchronous coordinates are not completely fixed since a residual coordinate freedom remains, what renders the interpretation of calculations in this gauge difficult.

*Longitudinal gauge*: This gauge is defined by the conditions  $B = E = 0$  and, in this gauge, coordinates are totally fixed since  $E = 0$  determines  $\xi$  uniquely. Using this result,  $B = 0$  allows to fix determines  $\xi^0$  without any uncertainty. We draw the important conclusion that in this gauge  $\phi$  and  $\psi$  coincide with the gauge invariant variables (3.15)  $\Phi$  and  $\Psi$  respectively which have a simple physical interpretation as the amplitudes of the metric perturbations in the usually so-called conformal-Newtonian coordinate system.

### 3.2.3 Equations for cosmological perturbations in EH gravity

In  $\Lambda$ CDM model within the metric formalism it is possible to obtain a second order differential equation for the growth of matter density perturbation. Let us previously define the density contrast  $\delta$  as follows:

$$\delta \equiv \frac{\delta\rho}{\rho_0} \equiv \frac{\rho - \rho_0}{\rho_0} \quad (3.17)$$

where  $\rho_0$  holds for the unperturbed mean cosmological energy density for a fluid and  $\rho$  for the perturbed energy density of the same cosmological fluid.

In the following, as was mentioned in the beginning of the chapter, the longitudinal gauge will be considered to perform our calculations. Thus, the flat FLRW  $D = 4$  metric tensor with scalar perturbations expressed in this gauge and by using conformal time  $\tau$  is written as:

$$ds^2 = a^2(\tau) [(1 + 2\Phi)d\tau^2 - (1 - 2\Psi)(dr^2 + r^2 d\Omega_2^2)] \quad (3.18)$$

where  $\Phi \equiv \Phi(\tau, \vec{x})$  and  $\Psi \equiv \Psi(\tau, \vec{x})$  are the well-known Bardeen's potentials [74]. From this metric, the first order perturbed standard Einstein equations are obtained:

$$\delta G^\mu{}_\nu = -8\pi G \delta T^\mu{}_\nu \quad (3.19)$$

Now that the metric with scalar perturbations is known,  $\delta G_\nu^\mu$  is straightforwardly determined through Einstein's tensor  $G_\nu^\mu$  definition. Therefore the first order perturbed standard Einstein equations, i.e., for EH gravity read:

$$\begin{aligned}\delta G_0^0 &= -6\mathcal{H}^2\Phi - 6\mathcal{H}\Psi' + 2\nabla^2\Psi = 8\pi G a^2 \delta T_0^0 \\ \delta G_i^0 &= (2\mathcal{H}\Phi + 2\Psi')_{,i} = 8\pi G a^2 \delta T_i^0 \\ \delta G_j^i &= (-4\mathcal{H}'\Phi - 2\mathcal{H}^2\Phi - 2\mathcal{H}\Phi' - 2\Psi'' - 4\mathcal{H}\Psi' - \nabla^2 D)\delta_j^i + D_{,j}{}^i = 8\pi G a^2 \delta T_j^i\end{aligned}\tag{3.20}$$

where  $D \equiv \Phi - \Psi$ , the prime denotes derivative with respect to conformal time  $\tau$  and the subindex  $,i$  is the usual derivative with respect to the  $i^{th}$ -spatial coordinate.

To study the growth rate of cosmological perturbations we shall consider models with conventional hydrodynamical matter described by a perfect fluid energy-momentum tensor as given in Section 1.2. It should be reminded that  $u^\mu \equiv dx^\mu/ds$  is the mean 4-velocity of the fluid. Unperturbed 4-velocity in FLRW conformal coordinates becomes:

$$u_{(0)}^\mu = \frac{1}{a(\tau)} (1, \vec{0})\tag{3.21}$$

and to first order in scalar perturbations, it can be shown that

$$u^\mu = a(\tau)^{-1}(1 - \Phi, \delta u^i)\tag{3.22}$$

where  $\delta u^i$  can be decomposed as follows:

$$\delta u^i = \bar{u}^i + v^i,\tag{3.23}$$

with  $\bar{u}^i$  and  $v^i$  being the solenoidal and irrotational components respectively. Note at this stage that  $\bar{u}^i$  only contributes to the vector perturbations but not to the scalar ones and  $v$  is usually referred to as the potential for velocity perturbations.

Taking into account the previous digression, the perturbed energy-momentum tensor components are proven to be:

$$\begin{aligned}\delta T_0^0 &= \delta\rho = \rho_0\delta \\ \delta T_j^i &= -(\delta P)\delta_j^i \\ \delta T_i^0 &= -\delta T_0^i = -(\rho_0 + P_0)\partial_i v\end{aligned}\tag{3.24}$$

with  $\delta P$  pressure fluctuation. Substituting expressions (3.24) in (3.20), they become

$$\begin{aligned}-3\mathcal{H}^2\Phi - 3\mathcal{H}\Psi' + \nabla^2\Psi &= 4\pi G a^2 \delta\rho \\ (\mathcal{H}\Phi + \Psi')_{,i} &= 4\pi G a^2 (\rho_0 + P_0) v_{,i} \\ (-2\mathcal{H}'\Phi - \mathcal{H}^2\Phi - \mathcal{H}\Phi' - \Psi'' - 2\mathcal{H}\Psi' - \frac{1}{2}\nabla^2 D)\delta_j^i + \frac{1}{2}D_{,j}{}^i &= -4\pi G a^2 \delta P \delta_j^i\end{aligned}\tag{3.25}$$

For  $i \neq j$  considered,  $D_{,j}^i = 0$  which in Fourier space means  $k_i k_j D_k = 0$  for any  $i, j$  values and then  $D$  is identically null and thus

$$\Phi(\tau, \vec{x}) \equiv \Psi(\tau, \vec{x}). \quad (3.26)$$

Such a result permits to simplify the previous equations (3.25) to become:

$$\begin{aligned} \nabla^2 \Phi - 3\mathcal{H}^2 \Phi - 3\mathcal{H}\Phi' &= 4\pi G a^2 \delta\rho \\ (a\Phi)_{,i}' &= 4\pi G a^3 (\rho_0 + P_0) v_{,i} \\ \Phi'' + 3\mathcal{H}\Phi' + (\mathcal{H}^2 + 2\mathcal{H}')\Phi &= 4\pi G a^2 \delta P. \end{aligned} \quad (3.27)$$

At this stage, a short digression about the pressure  $P$  dependence may be valuable: the pressure is, in principle, a quantity depending on energy density and entropy per baryon ratio. Thus, a pressure fluctuation  $\delta P$  can be expressed in terms of density and entropy perturbations as follows

$$\delta P = \left( \frac{\partial P}{\partial \rho} \right)_S \delta\rho + \left( \frac{\partial P}{\partial S} \right)_\rho \delta S \equiv c_S^2 \delta\rho + \left( \frac{\partial P}{\partial S} \right)_\rho \delta S \quad (3.28)$$

where  $c_S^2$  can be understood as the squared sound velocity of the fluid perturbations. In a single component perfect fluid with constant equation of state there are no entropy perturbations. However, if the perturbation description needs to include more than one component, entropy perturbations may be present.

In the following we shall restrict ourselves to adiabatic perturbations, i.e.  $\delta S = 0$  and therefore

$$\delta P = c_S^2 \delta\rho \quad (3.29)$$

and equation of state for the fluid will be considered constant, i.e.  $P = \omega\rho$  with  $\omega$  constant. Thus, perturbed and unperturbed content matter are assumed to have the same equation of state, i.e.  $\delta P/\delta\rho \equiv c_S^2 \equiv P_0/\rho_0$ , where  $c_S = 0$  for dust matter adiabatic perturbations.

With the previous assumptions, the equations (3.27) can be combined to obtain the growth rate  $\delta$  evolution in Fourier space <sup>2</sup>. For instance, for dust matter the resulting differential equation is

$$\delta'' + \mathcal{H} \frac{k^4 - 6\tilde{\rho}k^2 - 18\tilde{\rho}^2}{k^4 - \tilde{\rho}(3k^2 + 9\mathcal{H}^2)} \delta' - \tilde{\rho} \frac{k^4 + 9\tilde{\rho}(2\tilde{\rho} - 3\mathcal{H}^2) - k^2(9\tilde{\rho} - 3\mathcal{H}^2)}{k^4 - \tilde{\rho}(3k^2 + 9\mathcal{H}^2)} \delta = 0 \quad (3.30)$$

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<sup>2</sup>In the rest of the present chapter it must be understood that symbols  $\delta$ ,  $\Phi$ ,  $\Psi$  and  $v$  will hold for the Fourier corresponding quantities but the subindex  $k$  will be omitted in order to simplify the notation.

where  $\tilde{\rho} \equiv 4\pi G\rho_0 a^2 = -\mathcal{H}' + \mathcal{H}^2$  according to the background standard Einstein equations, as seen for instance from (2.15) setting  $P_0 \equiv 0$  and  $f(R)$  constant. We point out that in order to obtain the equation (3.30) it is not necessary to calculate the potentials  $\Phi$  and  $\Psi$  explicitly, but algebraic manipulations in the equations (3.27) are enough to get this result.

Some limits can be taken in the previous equation: for instance, it is of particular interest to consider those  $k$  modes whose wavelength is much smaller than the Hubble radius. These modes are known as sub-Hubble modes and are identified by the condition  $k \gg \mathcal{H}$  or equivalently  $k\tau \gg 1$ . In this approximation the equation (3.30) reduces to the well-known expression:

$$\delta'' + \mathcal{H}\delta' - 4\pi G\rho_0 a^2 \delta = 0. \quad (3.31)$$

In this regime and at early times, the matter energy density dominates over the cosmological constant term and it is easy to show that  $\delta$  solutions for (3.31) grow as  $a(\tau)$ . At late times (near today) the cosmological constant contribution is not negligible and thus the equation (3.31) does not admit any more power-law solutions of the type  $\delta \propto a(\tau)^\gamma$  for some  $\gamma$ . It is necessary in this case to assume an ansatz for  $\delta$ : one which works very well is that proposed in references [66] and [75] namely

$$\frac{\delta(a)}{a} = \exp \left[ \int_{a_i}^a (\Omega_M(a)^\gamma - 1) d\ln a \right] \quad (3.32)$$

where  $\Omega_M(a) \equiv \frac{\Omega_M \mathcal{H}_0^2}{a \mathcal{H}^2}$  and  $\Omega_M$  was defined in expression (2.10). Expression (3.32) turns out to fit with high precision the numerical solution for  $\delta$  with a constant parameter  $\gamma = 6/11$ .

### 3.3 Cosmological perturbations in $f(R)$ theories

#### 3.3.1 Perturbed Einstein equations in $f(R)$ theories

Using the perturbed metric (3.1) and the perturbed energy-momentum tensor (3.24), the first order perturbed equations for  $f(R)$  theories in the metric formalism, assuming that the background equations given in expression (1.10) in Chapter 1 hold, may be written as:

$$\begin{aligned} (1 + f_R)\delta G_\nu^\mu + ((R_{(0)})_\nu^\mu + \nabla^\mu \nabla_\nu - \delta_\nu^\mu \square) f_{RR} \delta R + [(\delta g^{\mu\alpha}) \nabla_\nu \nabla_\alpha - \delta_\nu^\mu (\delta g^{\alpha\beta}) \nabla_\alpha \nabla_\beta] f_R \\ - \left[ g_{(0)}^{\alpha\mu} (\delta \Gamma_{\alpha\nu}^\gamma) - \delta_\nu^\mu g_{(0)}^{\alpha\beta} (\delta \Gamma_{\beta\alpha}^\gamma) \right] \partial_\gamma f_R = -8\pi G \delta T_\nu^\mu \end{aligned} \quad (3.33)$$

where  $(R_{(0)})_\nu^\mu$  will denote here the Ricci tensor components corresponding to the unperturbed FLRW metric (2.2) in comoving coordinates whose trace provides the scalar

curvature already given in equation (2.8). Note that  $f(R)$  derivatives with respect to  $R_{(0)}$  have been expressed as usually, i.e.,  $f_R \equiv df(R)/dR|_{R_{(0)}}$ ,  $f_{RR} = d^2f(R)/dR^2|_{R_{(0)}}$  and again  $\square \equiv \nabla_\alpha \nabla^\alpha$  and  $\nabla$  is the usual covariant derivative with respect to the unperturbed FLRW metric. Notice also that unlike the ordinary EH case where Einstein's equations are second order, the equations (3.33) constitute a set of fourth order differential equations.

For the linearized modified Einstein equations, the components  $(00)$ ,  $(ii)$ ,  $(0i) \equiv (i0)$  and  $(ij)$ , where  $i, j = 1, 2, 3$ ,  $i \neq j$ , in Fourier space, read respectively:

$$(1 + f_R)[-k^2(\Phi + \Psi) - 3\mathcal{H}(\Phi' + \Psi') + (3\mathcal{H}' - 6\mathcal{H}^2)\Phi - 3\mathcal{H}'\Psi] + 3f'_R[\mathcal{H}(-3\Phi + \Psi) - \Psi'] = 2\tilde{\rho}\delta \quad (3.34)$$

$$(1 + f_R)[\Phi'' + \Psi'' + 3\mathcal{H}(\Phi' + \Psi') + 3\mathcal{H}'\Phi + (\mathcal{H}' + 2\mathcal{H}^2)\Psi] + f'_R(3\mathcal{H}\Phi - \mathcal{H}\Psi + 3\Phi') + f''_R(3\Phi - \Psi) = 2c_s^2\tilde{\rho}\delta \quad (3.35)$$

$$(1 + f_R)[\Phi' + \Psi' + \mathcal{H}(\Phi + \Psi)] + f'_R(2\Phi - \Psi) = -2\tilde{\rho}(1 + c_s^2)v \quad (3.36)$$

$$\Phi - \Psi = -\frac{f_{RR}}{1 + f_R}\delta R \quad (3.37)$$

where  $\delta R \equiv R - R_{(0)}$  is given by:

$$\delta R = -\frac{2}{a^2} \left[ 3\Psi'' + 6(\mathcal{H}' + \mathcal{H}^2)\Phi + 3\mathcal{H}(\Phi' + 3\Psi') - k^2(\Phi - 2\Psi) \right] \quad (3.38)$$

and  $\delta P = c_s^2\delta\rho$  has been again assumed.

By computing the covariant derivative with respect to the perturbed metric  $\tilde{\nabla}$  of the perturbed energy-momentum tensor  $\tilde{T}_\nu^\mu$ , we find the conservation equations:

$$\tilde{\nabla}_\mu \tilde{T}_\nu^\mu = 0 \quad (3.39)$$

which do not depend on  $f(R)$  explicitly. To first order, the equations (3.39) read

$$3\Psi'(1 + c_s^2) - \delta' + k^2(1 + c_s^2)v = 0 \quad (3.40)$$

and

$$\Phi + \frac{c_s^2}{1 + c_s^2}\delta + v' + \mathcal{H}v(1 - 3c_s^2) = 0 \quad (3.41)$$

for the temporal and spatial components respectively.

In a dust matter dominated universe, (3.40) and (3.41) can be combined to give

$$\delta'' + \mathcal{H}\delta' + k^2\Phi - 3\Psi'' - 3\mathcal{H}\Psi' = 0. \quad (3.42)$$

### 3.3.2 Equation for density perturbations in $f(R)$ theories

In this subsection, we are going to obtain the differential equation obeyed by  $\delta$  in a dust matter dominated universe when the longitudinal gauge and the metric formalism are considered to study first order scalar perturbations of  $f(R)$  theories.

To do so, let us consider equations (3.34) and (3.36) for a dust matter dominated universe, and combine them to express the potentials  $\Phi$  and  $\Psi$  in terms of  $\{\Phi', \Psi'; \delta, \delta'\}$  by means of algebraic manipulations. The resulting expressions are the following

$$\begin{aligned} \Phi &= \frac{1}{\mathcal{D}(\mathcal{H}, k)} \left\{ [3(1 + f_R)\mathcal{H}(\Psi' + \Phi') + f'_R \Psi' + 2\tilde{\rho}\delta](1 + f_R)(\mathcal{H} - f'_R) \right. \\ &\quad \left. + [(1 + f_R)(\Phi' + \Psi') + \frac{2\tilde{\rho}}{k^2}(\delta' - 3\Psi')][(1 + f_R)(-k^2 - 3\mathcal{H}') + 3f'_R \mathcal{H}] \right\} \quad (3.43) \end{aligned}$$

and

$$\begin{aligned} \Psi &= \frac{1}{\mathcal{D}(\mathcal{H}, k)} \left\{ [-3(1 + f_R)\mathcal{H}(\Psi' + \Phi') - 3f'_R \Psi' - 2\tilde{\rho}\delta][(1 + f_R)\mathcal{H} + 2f'_R] - [(1 + f_R) \right. \\ &\quad \left. \times (\Phi' + \Psi') + \frac{2\tilde{\rho}}{k^2}(\delta' - 3\Psi')][(1 + f_R)(-k^2 + 3\mathcal{H}' - 6\mathcal{H}^2) - 9\mathcal{H}f'_R] \right\} \quad (3.44) \end{aligned}$$

where

$$\mathcal{D}(\mathcal{H}, k) \equiv -6(1 + f_R)^2 \mathcal{H}^3 + 3\mathcal{H}[f_R'^2 + 2(1 + f_R)^2 \mathcal{H}'] + 3(1 + f_R)f'_R(-2\mathcal{H}^2 + k^2 + \mathcal{H}'). \quad (3.45)$$

The second step will be to derive equations (3.43) and (3.44) with respect to  $\tau$  and thus  $\Phi'$  and  $\Psi'$  may be rewritten algebraically in terms of  $\{\Phi'', \Psi''; \delta, \delta', \delta''\}$ . These last results can be substituted in equations (3.34) and (3.36) to obtain the potentials  $\Phi$  and  $\Psi$  just in terms of  $\{\Phi'', \Psi''; \delta, \delta', \delta''\}$ . So at this stage, let us summarize that we have been able to express the following quantities

$$\begin{aligned} \Phi &= \Phi(\Phi'', \Psi''; \delta, \delta', \delta'') \\ \Psi &= \Psi(\Phi'', \Psi''; \delta, \delta', \delta'') \\ \Phi' &= \Phi'(\Phi'', \Psi''; \delta, \delta', \delta'') \\ \Psi' &= \Psi'(\Phi'', \Psi''; \delta, \delta', \delta'') \end{aligned} \quad (3.46)$$

but we do not do here explicitly. By the previous expressions we mean that the functions on the l.h.s. depend on the functions inside the parenthesis on the r.h.s. in an algebraic way.

The natural reasoning at this point would be to try to obtain the potentials second derivatives  $\{\Phi'', \Psi''\}$  in terms of  $\{\delta, \delta', \delta''\}$  by an algebraic process. The chosen equations

to do so will be (3.42) and the first derivative of (3.37) with respect to  $\tau$ . In (3.42) it is necessary to substitute  $\Phi$  and  $\Psi'$  by the expressions obtained in (3.46) whereas (3.37) first derivative may be sketched as follows

$$\Phi' - \Psi' = -\frac{f_{RR}}{1+f_R}\delta R' + \left[ \frac{f_{RR}f'_R - f'_{RR}(1+f_R)}{(1+f_R)^2} \right] \delta R. \quad (3.47)$$

Before deriving, we are going to substitute  $\Psi''$  that appears on (3.37) by lower derivatives potentials  $\{\Phi, \Psi, \Phi', \Psi'\}$ ,  $\delta$  and its derivatives. To do so we consider (3.34) and (3.36) first derivatives with respect to  $\tau$  where the quantity  $v$  has been previously substituted by its expression in (3.40). Following this process we may express  $\Psi''$  as follows

$$\Psi'' = \Psi''(\Phi, \Psi, \Phi', \Psi'; \delta, \delta', \delta'') \quad (3.48)$$

and now substituting the previous result (3.48) in  $\delta R$  definition given by expression (3.37), we can derive equation (3.37) with respect to  $\tau$ . Solving a two algebraic equations system with equations (3.42) and (3.47) and introducing (3.46) we are able to express  $\{\Phi'', \Psi''\}$  in terms of  $\{\delta, \delta', \delta'', \delta'''\}$ .

$$\Phi'' = \Phi''(\delta, \delta', \delta'', \delta''') ; \Psi'' = \Psi''(\delta, \delta', \delta'', \delta'''). \quad (3.49)$$

Thus we substitute the results obtained in (3.49) straightforwardly in (3.46) in order to express  $\{\Phi, \Psi, \Phi', \Psi'\}$  in terms of  $\{\delta, \delta', \delta'', \delta'''\}$ . With the two potentials and their first derivatives as algebraic functions of  $\{\delta, \delta', \delta'', \delta'''\}$ , we perform the last step: We consider  $\Phi(\delta, \delta', \delta'', \delta''')$  and derive it with respect to  $\tau$ . The result should be equal to  $\Phi'(\delta, \delta', \delta'', \delta''')$  so we only need to express together these two results obtaining a fourth order differential equation for  $\delta$ . Once this fourth order differential equation has been solved we may go backwards and by using the results for  $\delta$  we obtain  $\{\Phi'', \Psi''\}$  from (3.49) as conformal time  $\tau$  functions. Analogously from (3.46) the behaviour of the potentials  $\{\Phi, \Psi\}$  and their first derivatives could be determined.

The resulting equation for  $\delta$  can be written as follows:

$$\beta_{4,f}\delta^{iv} + \beta_{3,f}\delta''' + (\alpha_{2,EH} + \beta_{2,f})\delta'' + (\alpha_{1,EH} + \beta_{1,f})\delta' + (\alpha_{0,EH} + \beta_{0,f})\delta = 0 \quad (3.50)$$

where the coefficients  $\beta_{i,f}$  ( $i = 0, \dots, 4$ ) involve terms with  $f'_R$  and  $f''_R$ , i.e. terms disappearing if the choice  $f_R$  constant is made. Equivalently,  $\alpha_{i,EH}$  ( $i = 0, 1, 2$ ) contain terms coming from EH term and the linear part in  $R_{(0)}$  of  $f(R)$ .

It is very useful to define the parameter  $\epsilon \equiv \mathcal{H}/k$  since it will allow us to perform a perturbative expansion of the previous coefficients  $\alpha$ 's and  $\beta$ 's in the sub-Hubble limit. Other dimensionless parameters which will be used are the following:

$$\kappa_i \equiv \frac{\mathcal{H}^{(i)}}{\mathcal{H}^{i+1}} \quad i = 1, 2, 3 \quad ; \quad f_i \equiv \frac{f_R^{(i)}}{\mathcal{H}^i f_R} \quad i = 1, 2. \quad (3.51)$$

where superindex  $(i)$  means  $i^{th}$  derivative with respect to time  $\tau$ . Expressing now the  $\alpha$ 's and  $\beta$ 's coefficients as parameter  $\epsilon$  expansions, we may write

$$\begin{aligned}\alpha_{i,\text{EH}} &= \sum_{k=1}^3 \alpha_{i,\text{EH}}^{(k)} \quad i = 0, 1, 2 \\ \beta_{i,f} &= \sum_{k=1}^7 \beta_{i,f}^{(k)} \quad i = 3, 4 \\ \beta_{i,f} &= \sum_{k=1}^8 \beta_{i,f}^{(k)} \quad i = 0, 1, 2\end{aligned}\tag{3.52}$$

where two consecutive terms in each series differ in a  $\epsilon^2$  factor. The expressions for the coefficients are too long to be written explicitly. Instead, in the following sections we shall show different approximated formulae which are proven to be useful in certain limits. As a consistency check, we find that, both in a matter dominated universe and in  $\Lambda$ CDM, all  $\beta$  coefficients are absent since  $f_1$  and  $f_2$  defined by expression (3.51) vanish identically. For these cases, equation (3.50) becomes equation (3.31) as expected. For instance, in the pure matter dominated case, coefficients  $\kappa$ 's are constant and they take the following values  $\kappa_1 = -1/2$ ,  $\kappa_2 = 1/2$ ,  $\kappa_3 = -3/4$  and  $\kappa_4 = 3/2$ .

Another important feature from our results is that, in general, without imposing  $|f_R| \ll 1$ , the quotient

$$\frac{\alpha_{1,\text{EH}} + \beta_{1,f}}{\alpha_{2,\text{EH}} + \beta_{2,f}}\tag{3.53}$$

is not always equal to  $\mathcal{H}$ . In fact only the quotients

$$\frac{\beta_{1,f}^{(1)}}{\beta_{2,f}^{(1)}} \quad \text{and} \quad \frac{\alpha_{1,\text{EH}}^{(1)}}{\alpha_{2,\text{EH}}^{(1)}}\tag{3.54}$$

are identically equal to  $\mathcal{H}$ . This last result, namely  $\alpha_{1,\text{EH}}^{(1)}/\alpha_{2,\text{EH}}^{(1)}$ , is in perfect agreement with  $\delta'$  coefficient in expression (3.31) when one is studying sub-Hubble modes in  $\Lambda$ CDM theory, i.e., when  $\beta_{i,f}$   $i = 0, \dots, 4$  are not present.

### 3.3.3 Evolution of sub-Hubble modes and the quasi-static approximation

We are interested in the possible effects on the growth of density perturbations once they enter the Hubble radius in the matter dominated era. In this case  $\mathcal{H} \ll k$  and therefore

the sub-Hubble limit  $\epsilon \ll 1$  can be considered. It can be seen that the  $\beta_{4,f}$  and  $\beta_{3,f}$  coefficients are suppressed by  $\epsilon^2$  with respect to  $\beta_{2,f}$ ,  $\beta_{1,f}$  and  $\beta_{0,f}$ , i.e., in this limit the equation for perturbations reduces to the following second order expression:

$$\delta'' + \mathcal{H}\delta' + \frac{(1+f_R)^5 \mathcal{H}^2 (-1+\kappa_1)(2\kappa_1-\kappa_2) - \frac{16}{a^8} f_{RR}^4 (\kappa_2-2) k^8 8\pi G \rho_0 a^2}{(1+f_R)^5 (-1+\kappa_1) + \frac{24}{a^8} f_{RR}^4 (1+f_R)(\kappa_2-2) k^8} \delta = 0 \quad (3.55)$$

where we have taken only the leading terms in the  $\epsilon$  expansion for both  $\alpha$  and  $\beta$  coefficients.

This expression can be compared with the one usually considered in the literature by performing the so-called quasi-static approximation, obtained after performing strong simplifications in the perturbed equations - (3.34), (3.35), (3.36), (3.37), (3.40) and (3.41) - by neglecting time derivatives of  $\Phi$  and  $\Psi$  potentials, (see [76]). Thus, for instance in references [77] and [78], the quasi-static approximation is given by:

$$\delta'' + \mathcal{H}\delta' - \frac{1 + 4 \frac{k^2}{a^2} \frac{f_{RR}}{1+f_R}}{1 + 3 \frac{k^2}{a^2} \frac{f_{RR}}{1+f_R}} \frac{4\pi G \rho_0 a^2}{1+f_R} \delta = 0. \quad (3.56)$$

This approximation has been nonetheless considered as too aggressive in [79] since neglecting time derivatives can remove important information about the evolution of perturbations.

Note also that in equation (3.55) there exists a difference in a power  $k^8$  between those terms coming from the  $f$ -part and those coming from the EH-part. This result differs from that in the quasi-static approximation where difference is in a power  $k^2$  according to expression (3.56).

In order to compare the evolution for both equations, we have considered a specific function

$$f_{test}(R) = -4 R^{0.63} \quad (3.57)$$

where  $H_0^2$  units have been used, which gives rise to a matter era followed by a late time accelerated phase with the correct deceleration parameter today. In fact this model belongs to Class **II**  $f(R)$  models presented in Section 2.4 and it is therefore cosmologically viable.

Initial conditions in the matter era were given at redshift  $z = 485$  where the EH-part was dominant. Results, for  $k = 0.2 h \text{ Mpc}^{-1}$  are presented in Figure 3.1. It can be seen that, as expected, both expressions give rise to the same evolutions at early times (large redshifts) where they also agree with the standard  $\Lambda$ CDM evolution. However, at late times the quasi-static approximation fails to describe the evolution of perturbations correctly.

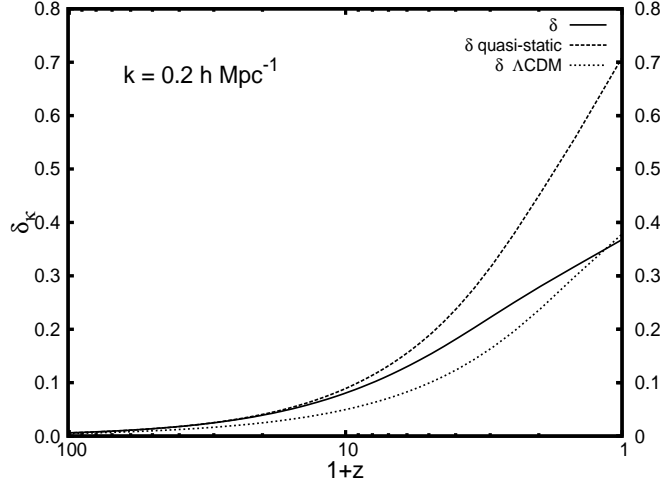


Figure 3.1:  $\delta_k$  with  $k = 0.2 h \text{ Mpc}^{-1}$  for  $f_{test}(R)$  model and  $\Lambda\text{CDM}$ . Both standard quasi-static evolution and equation (3.55) have been plotted in the redshift range from 100 to 0.

Notice that this  $f(R)$  model given by expression (3.57) satisfies all the viability conditions described in Section 1.6 except for the local gravity tests implemented by the condition 4 in that section. As is proven below, it is precisely this last condition namely

$$|f_R| \ll 1 \quad (3.58)$$

which in fact ensures the validity of the quasi-static approximation. Therefore we shall now restrict ourselves to  $f(R)$  models satisfying all the viability conditions, including  $|f_R| \ll 1$ .

In Appendix A.1 we have reproduced all the  $\alpha$ 's and the first four  $\beta$ 's coefficients for each  $\delta$  term in (3.50). When the sub-Hubble modes are studied and the condition  $|f_R| \ll 1$  is imposed, it can be shown that the dominant contributions are the first four  $\beta$  coefficients of the  $f$ -part plus the first  $\alpha$  coefficient of the EH-part for each term in equation (3.50). Thus in this case, the full differential equation (3.50) can be simplified as

$$c_4 \delta^{iv} + c_3 \delta''' + c_2 \delta'' + c_1 \delta' + c_0 \delta = 0 \quad (3.59)$$

where coefficients  $c_{0,1,\dots,4}$  are:

$$\begin{aligned} c_i &\equiv \lim_{|f_R| \ll 1} \left( \alpha_{i,\text{EH}}^{(1)} + \sum_{j=1}^4 \beta_{i,f}^{(j)} \right) \quad i = 0, 1, 2 \\ c_k &\equiv \lim_{|f_R| \ll 1} \sum_{j=1}^4 \beta_{k,f}^{(j)} \quad k = 3, 4 \end{aligned} \quad (3.60)$$

once the condition  $|f_R| \ll 1$  has been imposed on the corresponding  $\alpha$ 's and  $\beta$ 's contributions. These coefficients  $c_{0,\dots,4}$  have been explicitly written in Appendix A.2. We see that indeed in the sub-Hubble limit the  $c_4$  and  $c_3$  coefficients are negligible and the equation can be reduced to a second order expression. Moreover, for our approximated expression (3.59) it is true that  $c_1/c_2 \equiv \mathcal{H}$  as can be seen in Appendix A.2 straightforwardly.

From those expressions in Appendix A.2, the second order equation for  $\delta$  becomes

$$\delta'' + \mathcal{H}\delta' - \frac{4 \left[ \frac{6f_{RR}k^2}{a^2} + \frac{9}{4} \left( 1 - \sqrt{1 - \frac{8}{9} \frac{2\kappa_1 - \kappa_2}{-2 + \kappa_2}} \right) \right] \left[ \frac{6f_{RR}k^2}{a^2} + \frac{9}{4} \left( 1 + \sqrt{1 - \frac{8}{9} \frac{2\kappa_1 - \kappa_2}{-2 + \kappa_2}} \right) \right]}{3 \left[ \frac{6f_{RR}k^2}{a^2} + \frac{5}{2} \left( 1 - \sqrt{1 - \frac{24}{25} \frac{-1 + \kappa_1}{-2 + \kappa_2}} \right) \right] \left[ \frac{6f_{RR}k^2}{a^2} + \frac{5}{2} \left( 1 + \sqrt{1 - \frac{24}{25} \frac{-1 + \kappa_1}{-2 + \kappa_2}} \right) \right]} \times (1 - \kappa_1) \mathcal{H}^2 \delta = 0 \quad (3.61)$$

which can also be written as:

$$\delta'' + \mathcal{H}\delta' - \frac{4 \left( \frac{6f_{RR}k^2}{a^2} + \frac{9}{4} \right)^2 - \frac{81}{16} + \frac{9}{2} \frac{2\kappa_1 - \kappa_2}{-2 + \kappa_2}}{3 \left( \frac{6f_{RR}k^2}{a^2} + \frac{5}{2} \right)^2 - \frac{25}{4} + 6 \frac{-1 + \kappa_1}{-2 + \kappa_2}} (1 - \kappa_1) \mathcal{H}^2 \delta = 0. \quad (3.62)$$

Note that the quasi-static expression (3.56) is only recovered in the dust matter era (i.e. for  $\mathcal{H} = 2/\tau$ ) or for a pure  $\Lambda$ CDM evolution for the background dynamics. Nonetheless, in the considered limit  $|f_R| \ll 1$  it can be proven, using the background equations of motion, that

$$1 + \kappa_1 - \kappa_2 \approx 0 \quad (3.63)$$

and therefore  $2\kappa_1 - \kappa_2 \approx -2 + \kappa_2 \approx -1 + \kappa_1$  what allows to simplify the expression (3.62) to become (3.56). This is nothing but the fact that for viable models the background evolution resembles that of  $\Lambda$ CDM [31].

In other words, although for general  $f(R)$  functions the quasi-static approximation is not justified, for those viable  $f(R)$  functions describing the present phase of accelerated expansion and satisfying local gravity tests, it does give a correct description for the evolution of perturbations. This result has been here stated for the first time shedding some light about the controversy which remained about the validity of the quasi-static approximation.

### 3.3.4 Some proposed models

In order to illustrate the results obtained in the previous section, we propose two particular  $f(R)$  theories which allow us to determine - at least numerically - all the quantities involved in the calculations and therefore to obtain solutions for the equation (3.61).

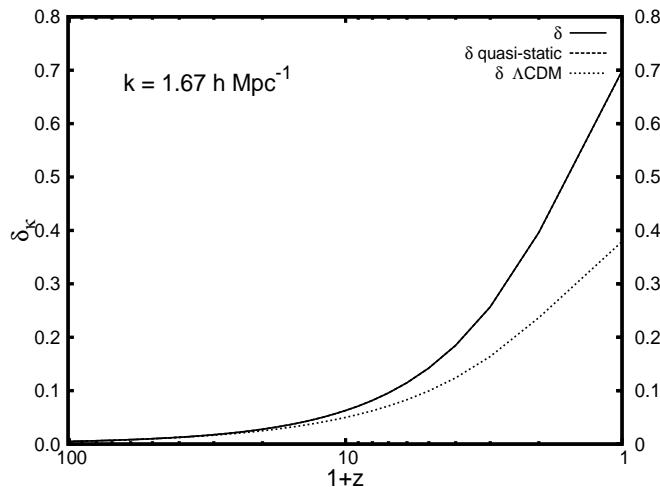


Figure 3.2:  $\delta_k$  with  $k = 1.67 h \text{ Mpc}^{-1}$  for  $f(R)$  model **A** evolving according to equation (3.61),  $\Lambda$ CDM and quasi-static approximation given by equation (3.56) in the redshift range from 100 to 0. The quasi-static evolution is indistinguishable from that coming from the equation (3.61), but diverges from  $\Lambda$ CDM behaviour as  $z$  decreases.

As was proven before, the background evolution for viable  $f(R)$  models resembles that of  $\Lambda$ CDM at low redshifts and that of a matter dominated universe at high redshifts. Nevertheless, the  $f(R)$  contribution gives the dominant contribution to the gravitational action (1.3) for small curvatures and therefore it may explain the cosmological acceleration. For the sake of concreteness, the models parameters have been fixed by imposing a deceleration parameter today  $q_0 \approx -0.6$ .

Thus, our first model **A** will be:

$$f(R) = c_1 R^p. \quad (3.64)$$

According to the results presented in references [55] and [80] viable models of this type belong to Class **II** introduced in Chapter 2. As was mentioned there, they both include matter dominated and late time accelerated eras provided the parameters satisfy  $c_1 < 0$  and  $0 < p < 1$ . We have chosen  $c_1 = -4.3$  and  $p = 0.01$  in  $H_0^2$  units. This choice does verify all the viability conditions, including  $|f_R| \ll 1$  today.

As a second model **B** we have chosen:

$$f(R) = \frac{1}{c_1 R^{e_1} + c_2} \quad (3.65)$$

with values  $c_1 = 2.5 \cdot 10^{-4}$ ,  $e_1 = 0.3$  and  $c_2 = -0.22$  also in the same units.

For each model, we compare our result (3.61) with the standard  $\Lambda$ CDM evolution and the quasi-static approximation (3.56) by plotting  $\delta$  evolution in Figures 3.2 and 3.3 for

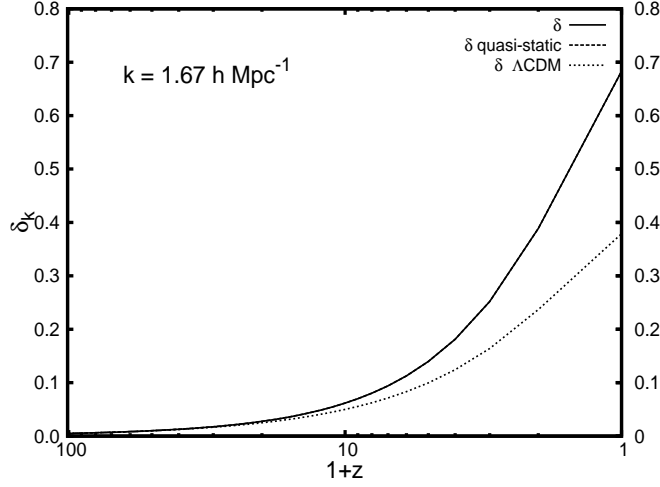


Figure 3.3:  $\delta_k$  with  $k = 1.67 h \text{ Mpc}^{-1}$  for  $f(R)$  model **B** evolving according to equation (3.61),  $\Lambda\text{CDM}$  and quasi-static evolution given by equation (3.56) in the redshift range from 100 to 0. The quasi-static evolution is indistinguishable from that coming from the equation (3.61), but diverges from  $\Lambda\text{CDM}$  behaviour as  $z$  decreases.

models **A** and **B** respectively. In both cases, the initial conditions are given at redshift  $z = 1000$  where  $\delta$  is assumed to behave as in a matter dominated universe, i.e.  $\delta_k(\tau) \propto a(\tau)$  with no  $k$ -dependence. We see that for both models, the quasi-static approximation gives a correct description for the evolution which clearly deviates from the  $\Lambda\text{CDM}$  case.

In Figure 3.4 the density contrast evaluated today was plotted as a function of  $k$  for both models. The growing dependence of  $\delta$  with respect to  $k$  is verified. This modified  $k$ -dependence with respect to the standard  $\Lambda\text{CDM}$  model could give rise to observable consequences in the matter power spectrum, as shown in [56, 72], and could be used to constrain or even to discard  $f(R)$  theories for cosmic acceleration as will be done in the next section.

### 3.4 A viable $f(R)$ model different from $\Lambda\text{CDM}$ ?

Some modified  $f(R)$  gravity models have recently been proposed (see for instance [58]) claiming to be cosmologically viable in spite of having a cosmological behaviour clearly distinguishable from  $\Lambda\text{CDM}$ . Contrary to already mentioned opinions which consider that self-consistent  $f(R)$  gravity models distinct from  $\Lambda\text{CDM}$  are almost ruled out, authors in [58] seem to claim that their proposed model would be cosmologically viable. We have shown [72] that although that model does satisfy some consistency conditions, precisely because of its departure from  $\Lambda\text{CDM}$  behaviour, it does not satisfy local gravity constraints

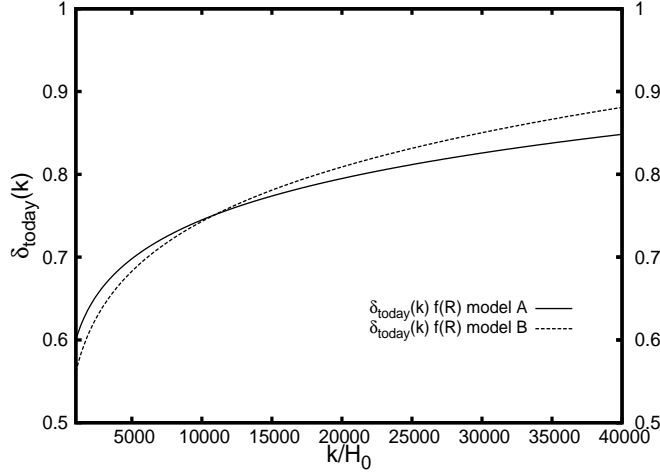


Figure 3.4: Scale dependence of  $\delta_k$  evaluated today ( $z = 0$ ) for  $k/H_0$  in the range from  $10^3$  to  $4 \cdot 10^4$ .

and, in addition, the predicted matter power spectrum conflicts with SDSS data provided in reference [69]. The proposed  $f(R)$  model reads

$$f(R) = -\alpha R_* \log \left( 1 + \frac{R}{R_*} \right). \quad (3.66)$$

This model does satisfy three of the usual viability conditions for  $f(R)$  theories provided in [31] and specified in Section 1.6. However, the model (3.66) does not satisfy the fourth of those conditions, namely,  $|f_R| \ll 1$  at recent epochs, imposed by local gravity tests [33] from solar system. Although it is still not clear what is the actual limit on this parameter, certain estimations give  $|f_R| < 10^{-6}$  today. This condition also ensures that the cosmological evolution at late times resembles that of  $\Lambda$ CDM. However, for the model (3.66),  $|f_R| \sim 0.2$  today for  $\alpha = 2$  and  $\Omega_M \sim 0.25$ .

If we are only interested in considering large scales, local gravity inconsistencies could be ignored, but still deviations from  $\Lambda$ CDM can have drastic cosmological consequences on the evolution of density perturbations, as discussed by several authors [56, 70, 79].

Thus, the linear evolution of matter density perturbations for sub-Hubble ( $k \gg \mathcal{H}$ ) modes in  $\Lambda$ CDM is given by equation (3.31). Notice that in this equation the evolution of the Fourier modes does not depend on  $k$ . This means that once the density contrast starts growing after matter-radiation equality, the mode evolution only changes the overall normalization of the matter power-spectrum, but not its shape.

However, in  $f(R)$  theories for sub-Hubble modes, as was thoroughly studied in the previous section, the corresponding equation reads as (3.55). Notice from this equation the  $k^8$  dependence in the  $\delta$  term which appears due to the fact that  $f_{RR} \neq 0$ . Moreover,

Redshift	EH term (in $\mathcal{H}_0$ units)	$f$ term (in $\mathcal{H}_0$ units)	$Rf_{RR}$
100	61.35	$1.90 \cdot 10^{-4}$	$1.45 \cdot 10^{-6}$
50	30.97	5.34	$1.12 \cdot 10^{-5}$
20	12.69	$2.86 \cdot 10^6$	$1.58 \cdot 10^{-4}$
0	-0.45	$1.01 \cdot 10^{17}$	0.079

Table 3.1: Values (in  $\mathcal{H}_0$  units) for both the first EH term and the first  $f$  term (the one proportional to  $k^8$ ) in the numerator of the  $\delta$  coefficient in equation (3.55) for this model (3.66). Different redshifts have been considered and the studied scale was  $k = 0.33 h \text{ Mpc}^{-1}$ . The EH term, which is  $k$  independent, cannot be ignored at high redshift. In fact, the redshift at which EH and  $f$  terms contributions (for this value of  $k$ ) equal is around  $z = 45$ . The strong suppression with the redshift of the  $f$ -part term – which is proportional to  $k^8$  in the  $\delta$  coefficient in equation (3.55) – comes from the rapid suppression of  $f_{RR}^4$  factor as the redshift increases. The dimensionless quantity  $Rf_{RR}$  has also been included in the last column of the table and it is seen to grow as the redshift decreases ( $z \rightarrow 0$ ).

a careful calculation of involved contributions in equation (3.55) shows that in the  $\delta$  term<sup>3</sup> there exist two contributions: one term is proportional to  $k^8$ , which is coming from the  $\beta$ 's contribution, i.e.  $f$ -part, and another term coming from  $\alpha$ 's contribution, i.e. EH-part. At high enough redshift the EH-part term dominates, whereas at low redshift the situation is reversed and the  $f$ -part term becomes dominant. This is the crucial point that explains why  $k$ -independent terms both in numerator and denominator cannot be straightforwardly removed from  $\delta$  coefficient in equation (3.55) but they have to be preserved for a correct sub-Hubble modes study. Thus for instance, for  $k = 0.33 h \text{ Mpc}^{-1}$ , we give explicit values in Table 3.1 for the terms in the numerator of the  $\delta$  coefficient in equation (3.55).

As a consequence the matter power-spectrum  $P_k^{f(R)}$  is further processed after equality and would differ today from the standard  $\Lambda\text{CDM}$  power spectrum  $P_k^{\Lambda\text{CDM}}$ . These two quantities would be related by a linear transfer function  $T(k)$  given by:

$$P_k^{f(R)} = T(k) P_k^{\Lambda\text{CDM}}. \quad (3.67)$$

This fact changes the shape of the matter power spectrum dramatically, as shown in Figure 3.5, where normalization to WMAP3 [81] was imposed. In this figure, SDSS data from luminous red galaxies [69] and the  $\Lambda\text{CDM}$  power spectrum from the linear perturbation theory with WMAP3 cosmological data [81] are also shown. Notice that  $\Lambda\text{CDM}$  gives an excellent fit to data with  $\chi^2 = 11.2$ , whereas for the  $f(R)$  theory  $\chi^2 = 178.9$ , i.e.  $13\sigma$  out. Even if the overall normalization is drastically reduced by a 20%, which is the present uncertainty over this parameter, the discrepancy would still remain at the  $7\sigma$  level. Actually, leaving the power spectrum normalization as a free parameter, the best fit would require a 32% normalization reduction and still would be  $4.8\sigma$  away as seen in Figure 3.5.

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<sup>3</sup>Let us consider the numerator of this term for simplicity. This does not mean any loss of generality since the denominator behaviour is completely analogous.

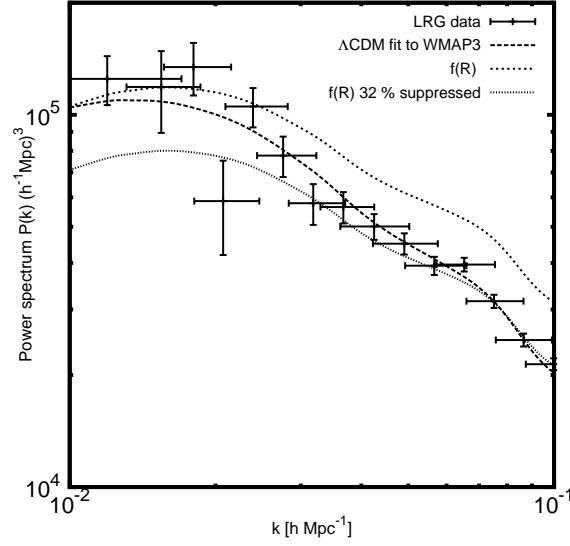


Figure 3.5: Linear matter power-spectra for  $\Lambda$ CDM and  $f(R)$  in [58] with  $\alpha = 2$ . Data were taken from SDSS [69].

The linear transfer function (3.67) for this model has been plotted in Figure 3.6 and it has been seen that  $T(k)$  follows with a nice fit a power law in the plotted interval  $T(k) = (k/k_{eq})^{0.19}$  where  $k_{eq} \simeq 10^{-2} h \text{ Mpc}^{-1}$  corresponds to the physical scale entering the Hubble radius when matter-radiation equality happened.

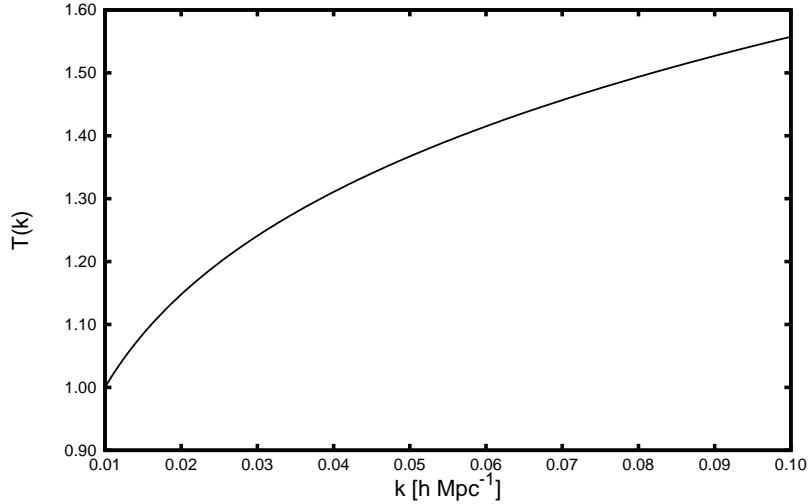


Figure 3.6: Transfer function  $T(k)$  for  $f(R)$  model given in equation (3.66) with  $\alpha = 2$  and  $\Omega_M \sim 0.25$ . The  $k$  dependence for  $T(k)$  with this parameters choice was seen to be  $T(k) \propto k^{0.19}$  in the plotted range of scales. Both the numerical  $T(k)$  and the power law proportional to  $k^{0.19}$  were plotted in scales  $k = 0.01 - 0.10 h \text{ Mpc}^{-1}$  and no difference was observed.

### 3.5 Conclusions

In this chapter we have studied the evolution of matter density perturbations in  $f(R)$  theories of gravity. Thus we have presented a completely general procedure to obtain the exact fourth order differential equation for the evolution of scalar perturbations in the longitudinal gauge. This expression is valid for any general  $f(R)$  theory and applicable at any scale  $k$ . If the EH gravitational action – both with and without cosmological constant – is considered in this general expression, well-known standard results are recovered.

We have also shown that for sub-Hubble modes, the obtained expression reduces to a second order differential equation. Hence, we have been able to compare this result with that obtained within the quasi-static approximation, widely used in the literature. Our research has explicitly shown that for arbitrary  $f(R)$  functions such an approximation is not justified.

However, if we limit ourselves to  $f(R)$  theories for which  $|f_R| \ll 1$  today, then the perturbative calculation for sub-Hubble modes requires to take into account, not only the first terms, but also higher-order terms in the  $\epsilon \equiv \mathcal{H}/k$  parameter. In that case, the resummation of such terms modifies the equation. Thus, this equation can be seen to be equivalent to the quasi-static case but only if the universe expands approximately as in a matter dominated phase or in a  $\Lambda$ CDM model. Finally, the fact that for  $f(R)$  models with  $|f_R| \ll 1$  the background behaves today precisely as that of  $\Lambda$ CDM makes the

quasi-static approximation correct in those cases.

We have finally applied our results to prove that no significant departure of  $f(R)$  theories from  $\Lambda$ CDM is allowed for those models that intend to be cosmologically viable according to recent data. In fact, the strong  $k$ -dependence appearing in the evolution of perturbations has allowed to rule out  $f(R)$  gravities which have recently been claimed to be cosmologically viable.



# Chapter 4

## Black holes in $f(R)$ theories

### 4.1 Introduction

We finish the exposition of our research on  $f(R)$  modified gravities by considering some aspects derived from the study of BHs in these theories.

Following the motivation already explained in Section 1.1,  $f(R)$  models may present BH solutions as GR and other alternative gravity theories do. Therefore it is quite natural to ask about BHs features in those gravitational theories since, on the one hand, some BHs signatures may be peculiar to Einstein's gravity and others may be robust features of all generally covariant theories of gravity. On the other hand, the results obtained may lead to rule out some models which will be in disagreement with expected physical results. For those purposes, research on thermodynamical quantities of BHs is of particular interest.

These attempts to detect particular signatures from these objects could be experimentally implemented, as was explained in Section 1.10, at the LHC in the coming years. Therefore the generation of mini BHs could provide important information about the correct underlying gravity theory.

Previous literature on  $f(R)$  theories [82] proved, by previously performing a conformal transformation in the gravitational action, that Schwarzschild solution is the only static spherically symmetric solution for an action of the form  $R + aR^2$  in  $D = 4$ . Also by using this conformal transformation, uniqueness theorems of spherically symmetric solutions for general polynomial actions in arbitrary dimensions were proposed in [83] (see also [84] for additional results and [85] for spherical solutions with sources).

Using the Euclidean action method [86, 87] in order to determine different thermodynamical quantities, anti-de Sitter ( $AdS$ ) BHs in  $f(R)$  models have been studied [88]. In

[89] the entropy of Schwarzschild-de Sitter BHs was calculated for some particular cosmologically viable models in vacuum and their cosmological stability was discussed.

BH properties have also been widely studied in other modified gravity theories: for instance, [90, 91] studied BHs in Einstein's theory with a Gauss-Bonnet term plus a cosmological constant. Different results were found depending on the dimension  $D$  and the sign of the constant horizon curvature  $k$ . For  $k = 0, -1$ , the Gauss-Bonnet term does not modify  $AdS$  BHs thermodynamics at all (only the horizon position is modified with respect to the EH theory) and BHs are not only locally thermodynamically stable, but also globally preferred. Nevertheless, for  $k = +1$  and  $D = 5$  (for  $D \geq 6$  thermodynamics is again essentially that for  $AdS$  BH) there exist some features not present in the absence of Gauss-Bonnet term. Gauss-Bonnet and/or Riemann squared interaction terms were studied in [92], where the authors concluded that in this case phase transitions may occur with  $k = -1$ .

Another approach is given by Lovelock gravities, which are free of ghosts and where the field equations contain no more than second derivatives of the metric. These theories were for instance studied in [93] and the corresponding entropy was evaluated.

The layout of this chapter is as follows: In Section 4.2 some generalities about BHs in  $f(R)$  theories such as several aspects of constant curvature solutions for static spherically symmetric cases with and without electric charge are presented. Then in Section 4.3 a perturbative approach around the EH action, with no previous imposition of constant curvature, is performed. There we shall find that up to second order in perturbations only BH solutions of the Schwarzschild-  $AdS$  type are present. Explicit expressions for the effective cosmological constant are obtained in terms of the  $f(R)$  function. To deal with such differential equations we have used the algebraic manipulation package **Mathematica** [94].

Finally, we shall consider in Section 4.4 the BHs thermodynamics in  $AdS$  space-time. There it will be found that this kind of solutions can only exist provided the theory satisfies  $R_0 + f(R_0) < 0$  where  $R_0$  holds for the constant curvature solution. Interestingly, this expression is proved to be related to the condition which guarantees the positivity of the effective Newton's constant in this type of theories. In addition, it also ensures that the thermodynamical properties in  $f(R)$  gravities are qualitatively similar to those of standard GR. Then, some consequences for local and global stability for some particular  $f(R)$  models will be provided in Section 4.5 and figures of thermodynamical regions will be shown in Section 4.6. The present chapter is finished by Section 4.7 remarking some conclusions of the presented results.

This chapter is mainly based upon the results that have been presented in references [26] and [95].

## 4.2 Constant curvature black-hole solutions

The most general static and spherically symmetric  $D \geq 4$  dimensional metric can be written as (see [96]):

$$ds^2 = e^{-2\Phi(r)} A(r) dt^2 - A^{-1}(r) dr^2 - r^2 d\Omega_{D-2}^2 \quad (4.1)$$

or alternatively

$$ds^2 = \lambda(r) dt^2 - \mu^{-1}(r) dr^2 - r^2 d\Omega_{D-2}^2 \quad (4.2)$$

where  $d\Omega_{D-2}^2$  is the metric on the  $S^{D-2}$  sphere. The identification  $\lambda(r) = e^{-2\Phi(r)} A(r)$  and  $\mu(r) = A(r)$  can be straightforwardly established. For obvious reasons, the  $\Phi(r)$  function is called the *anomalous redshift*: notice that a photon emitted at  $r$  with proper frequency  $\omega_0$  is measured at infinity with frequency  $\omega_\infty = \omega_0 e^{-\Phi(r)} \sqrt{A(r)}$ .

Since the metric is static, the scalar curvature  $R$  in  $D$  dimensions depends only on  $r$  and it is given, for the metric parametrization (4.1), by:

$$\begin{aligned} R(r) = & \frac{1}{r^2} \{ (D-2) [(D-3)(1-A(r)) + 2r(A(r)\Phi'(r) - A'(r))] \\ & + r^2 [3A'(r)\Phi'(r) - A''(r) - 2A(r)(\Phi'^2(r) - \Phi''(r))] \} \end{aligned} \quad (4.3)$$

where the prime denotes derivative with respect to  $r$ . At this stage it is interesting to ask about which are the most general static and spherically symmetric metric tensors with constant scalar curvature  $R_0$ . This metric tensor can be found by solving the equation  $R(r) = R_0$ . Thus, it is immediate to see that for a  $\Phi(r) = \Phi_0$  constant, this equation becomes

$$R(r) \equiv R_0 = \frac{1}{r^2} \left[ (D^2 - 5D + 6)(1 - A(r)) + rA'(r)(-2D + 4) - r^2 A''(r) \right] \quad (4.4)$$

whose general solution is

$$A(r) = 1 + a_1 r^{3-D} + a_2 r^{2-D} - \frac{R_0}{D(D-1)} r^2 \quad (4.5)$$

with  $a_1$  and  $a_2$  arbitrary integration constants. In fact, for the particular case  $D = 4$ ,  $R_0 = 0$  and  $\Phi_0 = 0$ , the metric can be written exclusively in terms of the function:

$$A(r) = 1 + \frac{a_1}{r} + \frac{a_2}{r^2}. \quad (4.6)$$

By establishing the identifications  $a_1 = -2GM$  and  $a_2 = Q^2$ , this solution corresponds to a Reissner-Nordström solution, i.e. a charged massive BH solution with mass  $M$  and charge  $Q$ . Further comments about this result will be made at the end of the section.

Now that the general static and spherically symmetric solution for constant curvature has been obtained as given by expression (4.5), let us insert the metric (4.1) into the general  $f(R)$  gravitational action (1.3), and let us also perform variations with respect to the functions  $A(r)$  and  $\Phi(r)$ , in order to find the corresponding modified Einstein equations for this parametrization. Thus, we obtain

$$(2 - D)(1 + f_R)\Phi'(r) - r [f_{RRR}R'(r)^2 + f_{RR}(\Phi'(r)R'(r) + R''(r))] = 0 \quad (4.7)$$

and

$$\begin{aligned} & 2rA(r)f_{RRR}R'(r)^2 + f_{RR}[2(D - 2)A(r)R'(r) + 2rA(r)R''(r) + A'(r)rR'(r)] \\ & + (1 + f_R)[2r(A(r)\Phi''(r) - A(r)\Phi'(r)^2) + 2(D - 2)A(r)\Phi'(r) - rA''(r) \\ & + A'(r)(2 - D + 3r\Phi'(r))] - r(R + f(R)) = 0 \end{aligned} \quad (4.8)$$

where as in previous chapters  $f_R \equiv df(R)/dR$ ,  $f_{RR} \equiv d^2f(R)/dR^2$  and  $f_{RRR} \equiv d^3f(R)/dR^3$ . The above equations look in principle quite difficult to solve. For this reason we shall firstly consider the case with constant scalar curvature  $R = R_0$  solutions. In this simple case the two previous equations reduce to:

$$(2 - D)(1 + f_R)\Phi'(r) = 0 \quad (4.9)$$

and

$$\begin{aligned} & R + f(R) + (1 + f_R) \left[ A''(r) + \frac{(D - 2)}{r} (A'(r) - 2A(r)\Phi'(r)) - 3A'(r)\Phi'(r) \right. \\ & \left. + 2A(r)(\Phi'(r)^2 - \Phi''(r)) \right] = 0 \end{aligned} \quad (4.10)$$

As was commented in the Section 1.5, the constant curvature solutions in vacuum of  $f(R)$  gravities are given by the equation (1.33) which can be rewritten as:

$$R_0 = \frac{Df(R_0)}{2(1 + f_R(R_0)) - D} \quad (4.11)$$

whenever  $2(1 + f_R(R_0)) \neq D$ . Thus from (4.9) one obtains  $\Phi'(r) = 0$  and then, for a constant scalar curvature  $R_0$ , the equation (4.10) becomes

$$R_0 + f(R_0) + (1 + f_R(R_0)) \left[ A''(r) + (D - 2)\frac{A'(r)}{r} \right] = 0. \quad (4.12)$$

By inserting expression (4.11) in the previous equation (4.12), we get

$$A''(r) + (D - 2)\frac{A'(r)}{r} = -\frac{2}{D}R_0. \quad (4.13)$$

This is a  $f(R)$ -independent linear second order inhomogeneous differential equation which can be easily integrated to give the general solution:

$$A(r) = c_1 + c_2 r^{3-D} - \frac{R_0}{D(D-1)} r^2 \quad (4.14)$$

which depends on two arbitrary constants  $c_1$  and  $c_2$ . However, this solution has no constant curvature in the general case since, as was found above, the constant curvature requirement demands  $c_1 = 1$ . Then for negative  $R_0$  this solution is basically the  $D$  dimensional generalization studied by Witten [86] of the BH in  $AdS$  space-time solution considered by Hawking and Page [87]. With the natural choice  $\Phi_0 = 0$ , that solution can be written as:

$$A(r) = 1 - \left( \frac{R_S}{r} \right)^{D-3} + \frac{r^2}{l^2} \quad (4.15)$$

where

$$R_S^{D-3} = \frac{16\pi G_D M}{(D-2)\mu_{D-2}} \quad (4.16)$$

with

$$\mu_{D-2} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)} \quad (4.17)$$

being the area of the  $D-2$  sphere,

$$l^2 \equiv -\frac{D(D-1)}{R_0} \quad (4.18)$$

is the asymptotic  $AdS$  space scale squared and  $M$  is the mass parameter usually found in the literature.

Thus we have concluded that the only static and spherically symmetric vacuum solutions with constant curvature of any  $f(R)$  gravity ( $R_0 < 0$  provided) is just the Hawking-Page BH in  $AdS$  space. However, this kind of solution is not the more general static and spherically symmetric metric with constant curvature as can be seen by comparison with the solutions found in expression (4.5). Therefore we have to conclude that there are constant curvature BH solutions that cannot be obtained as vacuum solutions of any  $f(R)$  theory.

Let us now consider the case of charged BHs in  $f(R)$  theories. For the sake of simplicity we shall limit ourselves to the  $D = 4$  case. The action of the theory will be now:

$$S_{f(R)\text{-Maxwell}} = \frac{1}{16\pi G} \int d^4x \sqrt{|g|} (R + f(R) - F_{\mu\nu} F^{\mu\nu}) \quad (4.19)$$

which is a generalization of the Einstein-Maxwell theory. The tensor  $F_{\mu\nu}$  is defined as:

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (4.20)$$

Considering an electromagnetic potential of the form:  $A_\mu = (V(r), \vec{0})$  and the static spherically symmetric metric (4.2), we find that the solution with constant curvature  $R_0$  reads:

$$\begin{aligned} V(r) &= -\frac{Q}{r}, \\ \lambda(r) &= \mu(r) = 1 - \frac{2GM}{r} + \frac{(1 + f_R(R_0))Q^2}{r^2} - \frac{R_0}{12}r^2. \end{aligned} \quad (4.21)$$

Notice that unlike the EH case, the contribution of the BH charge to the metric tensor is corrected by a  $(1 + f_R(R_0))$  factor.

### 4.3 Perturbative results

In the previous section we have considered static spherically symmetric solutions with constant curvature. In EH theory it is trivial to show that the only static and spherically symmetric solution possess constant scalar curvature. However, it is not guaranteed this to be the case in  $f(R)$  theories too. The problem of finding the general static spherically symmetric solution in arbitrary  $f(R)$  theories without imposing the constant curvature condition is in principle quite complicated. For that reason, we shall present in this section a perturbative analysis of the problem, assuming that the modified action given by the expression (1.3) is a small perturbation around EH theory.

The computation we are about to sketch is quite an involved one since, in order to calculate the solutions to a given order, it requires to introduce previous order results. To deal with such large equations we have used the algebraic manipulation package *Mathematica* [94].

Therefore let us consider a  $f(R)$  function of the form

$$f(R) = -(D - 2)\Lambda_D + \alpha g(R) \quad (4.22)$$

where  $\alpha \ll 1$  is a dimensionless parameter and  $g(R)$  is assumed to be analytic when expanded for perturbative solutions. Note that nonanalytic functions in  $\alpha$  are therefore excluded of this analysis.

By using the metric parametrization given by (4.2) the equations of motion become:

$$\begin{aligned} &\lambda(r)(1 + f_R) \{2\mu(r) [(D - 2)\lambda'(r) + r\lambda''(r)] + r\lambda'(r)\mu'(r)\} \\ &- 2\lambda(r)^2 \{2\mu(r) [(D - 2)R'(r)f_{RR} + rf_{RRR}R'(r)^2 + rR''(r)f_{RR}] + rR'(r)\mu'(r)f_{RR}\} \\ &- r\mu(r)\lambda'(r)^2(1 + f_R) + 2r\lambda(r)^2(R + f(R)) = 0 \end{aligned} \quad (4.23)$$

and

$$\begin{aligned} & \mu(r) \left\{ 2\lambda(r)R'(r) [2(D-2)\lambda(r) + r\lambda'(r)] f_{RR} + r(1 + f_R)(\lambda'(r)^2 - 2\lambda(r)\lambda''(r)) \right\} \\ & - \lambda(r)\mu'(r) [2(D-2)\lambda(r) + r\lambda'(r)] (1 + f_R) - 2r\lambda(r)^2(R + f(R)) = 0 \end{aligned} \quad (4.24)$$

where  $R \equiv R(r)$  is given by expression (4.3). Now the  $\lambda(r)$  and  $\mu(r)$  functions appearing in the metric (4.2) can be written as follows

$$\begin{aligned} \lambda(r) &= \lambda_0(r) + \sum_{i=1}^{\infty} \alpha^i \lambda_i(r) \\ \mu(r) &= \mu_0(r) + \sum_{i=1}^{\infty} \alpha^i \mu_i(r). \end{aligned} \quad (4.25)$$

Notice that  $\lambda_0(r)$  and  $\mu_0(r)$  are the unperturbed solutions for the EH action with cosmological constant given by

$$\begin{aligned} \mu_0(r) &= 1 + \frac{C_1}{r^{D-3}} - \frac{\Lambda_D}{(D-1)} r^2 \\ \lambda_0(r) &= C_2 \mu_0(r) \end{aligned} \quad (4.26)$$

which are the standard BH solutions in a  $D$  dimensional  $AdS$  spacetime ( $\Lambda_D < 0$  provided). Note that the factor  $C_2$  can be chosen by performing a coordinate  $t$  reparametrization so that both functions,  $\mu_0(r)$  and  $\lambda_0(r)$ , could be identified. For the moment, we shall keep the background solutions as given in (4.26) and we shall discuss the possibility of getting  $\lambda(r) = \mu(r)$  in the perturbative expansion later on.

By inserting the expressions (4.22) and (4.25) in the equations (4.23) and (4.24) we obtain the following

**First order equations:**

$$(D-3)\mu_1(r) + r\mu_1'(r) + \frac{2\Lambda_D g_R(R_0) - g(R_0)}{D-2} r^2 = 0 \quad (4.27)$$

$$\begin{aligned} & C_2 \left[ C_1(D-1)r^{3-D} - \Lambda_D r^2 + D-1 \right] g(R_0)r^2 + \left[ C_1(D-3)r^{3-D} + \frac{2\Lambda_D}{D-1} r^2 \right] \lambda_1(r) \\ & + C_2(D-2)(D-1) (\Lambda_D r^2 - D+3) \mu_1(r) \\ & + \left( 1 + C_1 r^{3-D} - \frac{\Lambda_D r^2}{D-1} \right) [2C_2(1-D)r^2 \Lambda_D g_R(R_0) + r\lambda_1'(r)] = 0 \end{aligned} \quad (4.28)$$

where  $g_R(R_0) \equiv dg(R)/dR|_{R=R_0}$  and whose solutions are:

$$\begin{aligned} \lambda_1(r) &= C_4(D-1)(D-2) + \frac{(C_1C_4 - C_2C_3)(D-2)(D-1)}{r^{D-3}} \\ &\quad - [C_4(D-2)\Lambda_D + C_2(g(R_0) - 2\Lambda_D g_R(R_0))]r^2 \end{aligned} \quad (4.29)$$

$$\mu_1(r) = \frac{C_3}{r^{D-3}} + \frac{(g(R_0) - 2\Lambda_D g_R(R_0))}{(D-2)(D-1)}r^2. \quad (4.30)$$

Up to second order in  $\alpha$ , we obtain the following

**Second order equations:**

$$(D-3)\mu_2(r) + r\mu_2'(r) + \frac{(g(R_0) - 2\Lambda_D g_R(R_0))}{D-2} \left( g_R(R_0) - \frac{2D}{D-2}\Lambda_D g_{RR}(R_0) \right) r^2 = 0 \quad (4.31)$$

$$\begin{aligned} &\left[ -C_1(D-3)r^{3-D} - \frac{2\Lambda_D r^2}{D-1} \right] \lambda_2(r) + C_2(D-2)(D-1) (-\Lambda_D r^2 + D-3) \mu_2(r) \\ &- \left( C_1 r^{4-D} + r - \frac{r^3 \Lambda_D}{D-1} \right) \lambda_2'(r) - C_3 C_4 (D-2)(D-1) (-\Lambda_D r^2 + D-3) r^{3-D} \\ &- C_2 [(D-1)(C_1 r^{3-D} + 1) - \Lambda_D r^2] \left[ 2\Lambda_D g_R(R_0)^2 + g(R_0) \left( \frac{2D\Lambda_D g_{RR}(R_0)}{D-2} - g_R(R_0) \right) \right] \\ &- \frac{4D\Lambda_D^2 g_R(R_0)g_{RR}(R_0)}{D-2} r^2 - C_4 [C_1(D-1)r^{3-D} + 2] [2\Lambda_D g_R(R_0) - g(R_0)] r^2 = 0 \end{aligned} \quad (4.32)$$

where  $g_{RR}(R_0) \equiv d^2g(R)/dR^2|_{R=R_0}$  and whose solutions are:

$$\begin{aligned} \lambda_2(r) &= C_6 + \frac{C_6 C_1 + (C_3 C_4 - C_2 C_5)(D-2)(D-1)}{r^{D-3}} \\ &\quad + \left[ -\frac{C_6 \Lambda_D}{D-1} + (g(R_0) - 2\Lambda_D g_R(R_0)) \left( C_4 + C_2 g_R(R_0) - \frac{2C_2 D \Lambda_D g_{RR}(R_0)}{D-2} \right) \right] r^2 \end{aligned} \quad (4.33)$$

$$\mu_2(r) = \frac{C_5}{r^{D-3}} + \frac{(g(R_0) - 2\Lambda_D g_R(R_0)) (2D\Lambda_D g_{RR}(R_0) - (D-2)g_R(R_0))}{(D-2)^2(D-1)} r^2. \quad (4.34)$$

Further orders in  $\alpha^{3,4,\dots}$  can be obtained by inserting the previous results in the subsequent differential equations to get  $\{\lambda_{3,4,\dots}(r), \mu_{3,4,\dots}(r)\}$ . Nevertheless, these equations become increasingly complicated to be explicitly solved.

Notice that from the obtained results up to second order in  $\alpha$ , the corresponding metric has constant scalar curvature for any value of the parameters  $C_{1,2,\dots,6}$ . As a matter of fact, this metric is nothing but the corresponding one to the standard Schwarzschild-*AdS* geometry, and can be easily rewritten in the usual form by making a trivial time reparametrization as follows:

$$\begin{aligned}\bar{\lambda}(r) &\equiv \lambda(r) [-C_2(D^2 + 3D - 2) + C_4(D^2 - 3D + 2)\alpha + C_6\alpha^2 + \mathcal{O}(\alpha^3)] \\ \bar{\mu}(r) &\equiv \mu(r).\end{aligned}\tag{4.35}$$

Therefore, at least up to second order, the only static, spherically symmetric solutions which are analytical in  $\alpha$  are the standard Schwarzschild-*AdS* space-times.

### 4.3.1 General expression to arbitrary order for constant curvature

Let us now assume from the very beginning that the solutions for the equations (4.23) and (4.24) belong to Schwarzschild-*AdS* BH type at any order in the  $\alpha$  expansion. Thus we can write ( $J > 0$  provided):

$$\lambda(r) \equiv \mu(r) = 1 - \left(\frac{\bar{R}_S}{r}\right)^{D-3} + Jr^2\tag{4.36}$$

as solutions for the modified Einstein equations (4.23) and (4.24) derived when the modification in the gravitational Lagrangian is given by expression (4.22). If we expand the quantities  $\bar{R}_S$  and  $J$  in terms of parameter  $\alpha$  we get:

$$\begin{aligned}\bar{R}_S &\equiv R_S + \sum_{i=1}^{\infty} C_i \alpha^i \\ J &\equiv -\frac{\Lambda_D}{(D-1)} + \sum_{i=1}^{\infty} J_i \alpha^i\end{aligned}\tag{4.37}$$

where  $R_S$  and  $C_i$  are arbitrary constants. The  $J_i$  coefficients can be determined from expression (1.33), which here becomes:

$$R - (D-2)\Lambda_D + \alpha g(R) + 2(D-1)J(1 + \alpha g'(R)) = 0\tag{4.38}$$

with  $R = -D(D-1)J$  is the obtained result when calculating  $R$  with functions given by (4.36). Expanding equation (4.38) in powers of  $\alpha$  it is possible to find a recurrence equation for the  $J_i$  coefficients, namely for the  $J_l$  (with  $l > 0$ ) coefficient, we find:

$$\begin{aligned} & (2-D)(D-1)J_l + \sum_{i=0}^{l-1} \sum_{\text{cond.1}} \frac{1}{i_1! i_2! \dots i_{l-1}!} (J_1)^{i_1} (J_2)^{i_2} \dots (J_{l-1})^{i_{l-1}} g^{(i)}(R_0) \\ & + 2(D-1) \sum_{k=0}^{l-1} J_k \sum_{i=0}^{l-k-1} \sum_{\text{cond.2}} \frac{1}{i_1! i_2! \dots i_{l-k-1}!} (J_1)^{i_1} (J_2)^{i_2} \dots (J_{l-k-1})^{i_{l-k-1}} g^{(i+1)}(R_0) = 0 \end{aligned} \quad (4.39)$$

with  $R_0 = -D(D-1)J_0 \equiv D\Lambda_D$  and  $g^{(i)}(R_0) \equiv d^{(i)}g(R)/dR^{(i)}|_{R=R_0}$ . In the previous recurrence relation, the first sum is done under the condition 1 given by:

$$\sum_{m=1}^{l-1} i_m = i, \quad i_m \in \mathbb{N} \cup \{0\} \quad \text{and} \quad \sum_{m=1}^{l-1} m i_m = l-1 \quad (4.40)$$

and the second one under the condition 2:

$$\sum_{m=1}^{l-k-1} i_m = i, \quad i_m \in \mathbb{N} \cup \{0\} \quad \text{and} \quad \sum_{m=1}^{l-k-1} m i_m = l-k-1. \quad (4.41)$$

For instance we have:

$$\begin{aligned} J_1 &= \frac{\mathcal{A}(g; D, \Lambda_D)}{(D-2)(D-1)} \\ J_2 &= -\frac{\mathcal{A}(g; D, \Lambda_D)[(D-2)g_R(R_0) - 2D\Lambda_D g_{RR}(R_0)]}{(D-2)^2(D-1)} \end{aligned} \quad (4.42)$$

where  $\mathcal{A}(g; D, \Lambda_D) \equiv g(R_0) - 2\Lambda_D g_R(R_0)$

Now we can consider the possibility of removing  $\Lambda_D$  from the gravitational Lagrangian (4.22) from the very beginning and still getting an  $AdS$  BH solution with an effective cosmological constant depending on  $g(R)$  and its derivatives evaluated at  $R_0 \equiv 0$ . In this case the results, order by order in  $\alpha$  up to order  $\alpha^2$ , are:

$$\begin{aligned} J_0(\Lambda_D = 0) &= 0, \\ J_1(\Lambda_D = 0) &= \frac{g(0)}{(D-2)(D-1)}, \\ J_2(\Lambda_D = 0) &= -\frac{g(0)g_R(0)}{(D-2)(D-1)}. \end{aligned} \quad (4.43)$$

As we see, in the context of  $f(R)$  gravities, it is therefore possible to have a BH in an  $AdS$

asymptotic space-time even if the initial cosmological constant  $\Lambda_D$  vanishes but  $\alpha g(0) > 0$  and  $\alpha$  small enough as considered in the explained reasoning.

To end this section we can summarize by saying that in the context of  $f(R)$  gravities the only spherically symmetric and static solutions in the general case (without imposing constant curvature) in perturbation theory up to second order are the standard BHs in  $AdS$  space. However, the possibility of having static and spherically symmetric solutions with nonconstant curvature cannot be excluded in the case of  $f(R)$  functions which are not analytical in  $\alpha$ .

## 4.4 Black-hole thermodynamics

In order to consider the different thermodynamic quantities for the  $f(R)$  BHs in  $AdS$  space-time, we start from the temperature. In principle, there are two different ways of introducing this quantity for the kind of solutions that we are considering here. Firstly we can use the definition coming from Euclidean quantum gravity [97]. In this case one introduces the Euclidean time  $\tau = it$  and the Euclidean metric  $ds_E^2$  is defined as:

$$-ds_E^2 = -d\sigma^2 - r^2 d\Omega_{D-2}^2 \quad (4.44)$$

where:

$$d\sigma^2 = e^{-2\Phi(r)} A(r) d\tau^2 + A^{-1}(r) dr^2. \quad (4.45)$$

The metric corresponds only to the region  $r > r_H$  where  $r_H$  is the outer horizon position with  $A(r_H) = 0$ . Expanding  $d\sigma^2$  near  $r_H$  we have:

$$d\sigma^2 = e^{-2\Phi(r_H)} A'(r_H) \rho d\tau^2 + \frac{d\rho^2}{A'(r_H) \rho} \quad (4.46)$$

where  $\rho = r - r_H$ . Now we introduce the new coordinates  $\tilde{R}$  and  $\theta$  defined as:

$$\theta = \frac{1}{2} e^{-\Phi(r_H)} A'(r_H) \tau \quad ; \quad \tilde{R} = 2 \sqrt{\frac{\rho}{A'(r_H)}} \quad (4.47)$$

so that:

$$d\sigma^2 = \tilde{R}^2 d\theta^2 + d\tilde{R}^2. \quad (4.48)$$

According to the Euclidean quantum gravity prescription,  $\tau$  coordinate in expression (4.46) is included in the interval defined by 0 and  $\beta_E = 1/T_E$ . On the other hand, in order to avoid conical singularities,  $\theta$  must run between 0 and  $2\pi$ . Thus it is found that

$$T_E = \frac{1}{4\pi} e^{-\Phi(r_H)} A'(r_H). \quad (4.49)$$

Another possible definition of temperature was firstly proposed in [98] stating that temperature can be given in terms of the horizon gravity  $\mathcal{K}$  as:

$$T_{\mathcal{K}} \equiv \frac{\mathcal{K}}{4\pi} \quad (4.50)$$

where  $\mathcal{K}$  is given by:

$$\mathcal{K} = \lim_{r \rightarrow r_H} \frac{\partial_r g_{tt}}{\sqrt{|g_{tt}g_{rr}|}}. \quad (4.51)$$

Then it is straightforward to find:

$$T_{\mathcal{K}} = T_E. \quad (4.52)$$

Therefore, both definitions give the same result for this kind of metric tensors. Notice also that in any case the temperature depends only on the behaviour of the metric near the horizon but it is independent from the gravitational action. By this we mean that different actions having the same solutions have also the same temperature. This is not the case for other thermodynamic quantities as we shall see later. Taking into account the results in previous sections for Schwarzschild-  $AdS$  BHs, we shall concentrate for simplicity only on those solutions, i.e. for a metric as (4.2) where  $A(r)$  is given by expression (4.15) and  $\Phi = 0$  is adopted as a natural choice.

Then, both definitions of temperature lead to:

$$\beta = \frac{1}{T} = \frac{4\pi l^2 r_H}{(D-1)r_H^2 + (D-3)l^2}. \quad (4.53)$$

Notice that the temperature is a function of  $r_H$  only, i.e. it depends only on the BH size. In the limit  $r_H$  going to zero the temperature diverges as  $T \sim 1/r_H$  and for  $r_H$  going to infinity  $T$  grows linearly with  $r_H$ . Consequently  $T$  has a minimum at:

$$r_{H0} = l \sqrt{\frac{D-3}{D-1}} \quad (4.54)$$

corresponding to a temperature:

$$T_0 = \frac{\sqrt{(D-1)(D-3)}}{2\pi l}. \quad (4.55)$$

The existence of this minimum was established in [87] for  $D = 4$  by Hawking and Page long time ago and it is well-known. More recently Witten extended this result to higher dimensions [86]. This minimum in the temperature is important in order to set the regions

with different thermodynamic behaviours and stability properties. For  $D = 4$ , an exact solution can be found for  $r_H$ :

$$r_H = l \frac{\left(9\frac{R_S}{l} + \sqrt{12 + 81\frac{R_S^2}{l^2}}\right)^{2/3} - 12^{1/3}}{18^{1/3} \left(9\frac{R_S}{l} + \sqrt{12 + 81\frac{R_S^2}{l^2}}\right)^{1/3}}. \quad (4.56)$$

Thus, in the  $R_S \ll l$  limit, we find  $r_H \simeq R_S$ , whereas in the opposite case  $l \ll R_S$ , we get  $r_H \simeq (l^2 R_S)^{1/3}$ . For the particular case  $D = 5$ ,  $r_H$  can also be exactly found to be:

$$r_H^2 = \frac{l^2}{2} \left( \sqrt{1 + \frac{4R_S^2}{l^2}} - 1 \right) \quad (4.57)$$

which goes to  $R_S^2$  for  $R_S \ll l$  and to  $lR_S$  for  $l \ll R_S$ . Notice that for any  $T > T_0$ , we have two possible BH sizes: one corresponding to the small BH phase with  $r_H < r_{H0}$  and the other corresponding to the large BH phase with  $r_H > r_{H0}$ .

In order to compute the remaining thermodynamic quantities, the Euclidean action

$$S_E = -\frac{1}{16\pi G_D} \int d^D x \sqrt{g_E} (R + f(R)) \quad (4.58)$$

is considered. Extending to the  $f(R)$  theories the computation by Hawking and Page [87] and Witten [86], we evaluate the gravitational Lagrangian in the Schwarzschild- $AdS$  scalar curvature solution times the difference between the  $AdS$  space-time volume minus the BH space-time volume. Thus, we may write:

$$\Delta S_E = -\frac{R_0 + f(R_0)}{16\pi G_D} \Delta V \quad (4.59)$$

where  $R_0 = -D(D-1)/l^2$  and  $\Delta V$  is the volume difference between  $AdS$  and BH solutions, which is given by:

$$\Delta V = \frac{\beta \mu_{D-2}}{2(D-1)} (l^2 r_H^{D-3} - r_H^{D-1}). \quad (4.60)$$

Notice that from these expressions, it is straightforward to obtain the free energy  $F$  since  $\Delta S_E = \beta F$  and therefore

$$\Delta S_E = -\frac{(R_0 + f(R_0))\beta \mu_{D-2}}{36\pi(D-1)G_D} (l^2 r_H^{D-3} - r_H^{D-1}) = \beta F. \quad (4.61)$$

We see that provided  $-(R_0 + f(R_0)) > 0$ , which is the usual case in EH gravity, one has  $F > 0$  for  $r_H < l$  and  $F < 0$  for  $r_H > l$ . The temperature corresponding to the horizon radius  $r_H = l$  will be denoted  $T_1$  and it is given by:

$$T_1 = \frac{D-2}{2\pi l}. \quad (4.62)$$

Notice that for  $D > 2$  we have  $T_0 < T_1$ .

On the other hand, the total thermodynamical energy may now be obtained as:

$$E = \frac{\partial \Delta S_E}{\partial \beta} = -\frac{(R_0 + f(R_0))Ml^2}{2(D-1)} \quad (4.63)$$

where  $M$  is the mass defined in (4.16). This is one of the possible definitions for the BH energy for  $f(R)$  theories, see for instance [99] for a more general discussion. For the EH action with nonvanishing cosmological constant one has  $f(R) = -(D-2)\Lambda_D$  and then it is immediate to find  $E = M$ . However, this is not the case for general  $f(R)$  actions. Notice that positive energy in  $AdS$  space-time requires  $R_0 + f(R_0) < 0$ . Now the entropy  $S$  can be obtained from the well-known relation:

$$S = \beta E - \beta F. \quad (4.64)$$

Then one gets:

$$S = -\frac{(R_0 + f(R_0))l^2 A_{D-2}(r_H)}{8(D-1)G_D} \quad (4.65)$$

where  $A_{D-2}(r_H)$  is the horizon area given by  $A_{D-2}(r_H) \equiv r_H^{D-2} \mu_{D-2}$ . Notice that once again positive entropy requires  $R_0 + f(R_0) < 0$ . For the EH action with nonvanishing cosmological constant one has  $f(R) = -(D-2)\Lambda_D$  one has  $R_0 + f(R_0) = -2(D-1)/l^2$  and then the Hawking-Bekenstein result [100]

$$S = \frac{A_{D-2}(r_H)}{4G_D}. \quad (4.66)$$

is recovered. Finally, one can compute the heat capacity  $C$  which can be written as:

$$C = \frac{\partial E}{\partial T} = \frac{\partial E}{\partial r_H} \frac{\partial r_H}{\partial T}. \quad (4.67)$$

Then it is easy to find

$$C = \frac{-(R_0 + f(R_0))(D-2)\mu_{D-2}r_H^{D-2}l^2}{8G_D(D-1)} \frac{(D-1)r_H^2 + (D-3)l^2}{(D-1)r_H^2 - (D-3)l^2}. \quad (4.68)$$

For the already mentioned case of the EH action with nonvanishing cosmological constant one finds:

$$C = \frac{(D-2)\mu_{D-2}r_H^{D-2}}{4G_D} \frac{(D-1)r_H^2 + (D-3)l^2}{(D-1)r_H^2 - (D-3)l^2}. \quad (4.69)$$

In the Schwarzschild limit  $l \rightarrow \infty$ , this formula gives:

$$C \simeq -\frac{(D-2)\mu_{D-2}r_H^{D-2}}{4G_D} < 0 \quad (4.70)$$

which is the well-known negative result for standard BHs of this type. In the general case, assuming like in the EH case  $(R_0 + f(R_0)) < 0$ , one finds  $C > 0$  for  $r_H > r_{H0}$  (the large BH region) and  $C < 0$  for  $r_H < r_{H0}$  (the small BH region). For  $r_H \sim r_{H0}$  ( $T$  close to  $T_0$ )  $C$  is divergent. Notice that in EH gravity,  $C < 0$  necessarily implies  $F > 0$  since  $T_0 < T_1$ .

In any case, for  $f(R)$  theories with  $R_0 + f(R_0) < 0$ , we have found an scenario similar to the one described in full detail by Hawking and Page in [87] long time ago for the EH case.

For  $T < T_0$ , the only possible state of thermal equilibrium in an  $AdS$  space is pure radiation with negative free energy and there is no stable BH solutions. For  $T > T_0$  we have two possible BH solutions: the small (and light) BH and the large (heavy) BH. The small one has negative heat capacity and positive free energy as the standard Schwarzschild BH. Therefore this last configuration is unstable under Hawking radiation decay. For the large BH we have two possibilities: if  $T_0 < T < T_1$  then both the heat capacity and the free energy are positive and the BH will decay by tunnelling into radiation, but if  $T > T_1$  then the heat capacity is still positive but the free energy becomes negative. In this case the free energy of the heavy BH will be less than that of pure radiation. Then pure radiation will tend to tunnel or to collapse to the BH configuration in equilibrium with thermal radiation.

In arbitrary  $f(R)$  theories one could in principle consider the possibility of having  $R_0 + f(R_0) > 0$ . However, in this case both the energy and the entropy, given by expressions (4.63) and (4.65) respectively, would be negative and therefore in such theories the  $AdS$  BH solutions would be unphysical. Therefore,  $R_0 + f(R_0) < 0$  can be regarded as a necessary condition for  $f(R)$  theories in order to support the existence of  $AdS$  BH solutions. Using (1.33), this condition implies  $1 + f_R(R_0) > 0$ . Let us remind that this condition has a clear physical interpretation in  $f(R)$  gravities already presented as the condition **2** in Section 1.5.

## 4.5 Particular examples

In this section we are going to consider several  $f(R)$  models in order to calculate the heat capacity  $C$  and the free energy  $F$  since, as was explained in the previous section, these are the relevant thermodynamical quantities for local and global stability of BHs. For these particular models,  $R_0$  can be calculated exactly by using the relation (1.33) with  $R = R_0$ . In the following we will fix the  $D$  dimensional Schwarzschild radius in expression (4.16) as  $R_S^{D-3} = 2$  for simplicity.

The models that we consider in this section have been previously studied in the lite-

rature, but attention there was drawn in studying their cosmological viability according to conditions provided in Section 1.5. Here we draw our attention on thermodynamics for Schwarzschild- $AdS$  BH solutions for these  $f(R)$  gravities.

Both free energy and heat capacity signs are studied for the different values that parameters which appear in these  $f(R)$  functions take. Once these signs are known, both local or global thermodynamical stability can be determined for these  $f(R)$  theories following the reasoning explained at the end of the previous section. The considered models are:

**Model I:**  $f(R) = \alpha(-R)^\beta$

This model belongs to Class **II** of  $f(R)$  models presented in Section 2.4 if parameters satisfy  $\alpha < 0$  and  $0 < \beta < 1$  as seen from Table 2.1. and it could be therefore cosmologically viable.

Substituting in expression (1.33) for arbitrary dimensions we get

$$R \left[ \left( 1 - \frac{2}{D} \right) - \alpha(-R)^{\beta-1} \left( 1 - \frac{2}{D} \beta \right) \right] = 0. \quad (4.71)$$

Since we are only considering nonvanishing curvature solutions, then we find:

$$R_0 = - \left[ \frac{2-D}{(2\beta-D)\alpha} \right]^{1/(\beta-1)}. \quad (4.72)$$

Since  $D$  is assumed to be larger than 2, the condition  $(2\beta-D)\alpha < 0$  provides well defined scalar curvatures  $R_0$ . Thus, two separated regions have to be studied: Region 1  $\{\alpha < 0, \beta > D/2\}$  and Region 2  $\{\alpha > 0, \beta < D/2\}$ . For this model we also get

$$1 + f_R(R_0) = \frac{D(\beta-1)}{2\beta-D}. \quad (4.73)$$

Notice that in Region 1,  $1 + f_R(R_0) > 0$  for  $D > 2$ , since in this case  $\beta > 1$  is straightforwardly accomplished. In Region 2, we find that for  $D > 2$ , the requirement  $R_0 + f(R_0) < 0$ , i.e.  $1 + f_R(R_0) > 0$ , fixes  $\beta < 1$ , since this is the most stringent constraint over the parameter  $\beta$  in this region. Therefore the physical space of parameters in Region 2 is restricted to be  $\{\alpha > 0, \beta < 1\}$ .

In Figures 4.1, 4.2 and 4.3 we plot the physical regions in the parameter space  $(\alpha, \beta)$  corresponding to the different signs of  $(C, F)$ .

**Model II:**  $f(R) = -(-R)^\alpha \exp(q/R) - R$

This model may also belong to Class II of  $f(R)$  models presented in Section 2.4 if  $\alpha = 1$  as seen from Table 2.1 and could be therefore cosmologically viable.

In this case, a vanishing curvature solution appears provided  $\alpha > 1$ . In addition, we also have:

$$R_0 = \frac{2q}{2\alpha - D}. \quad (4.74)$$

To get  $R_0 < 0$  the condition  $q(2\alpha - D) < 0$  must hold and two separated regions will be studied: Region 1  $\{q > 0, \alpha < D/2\}$  and Region 2  $\{q < 0, \alpha > D/2\}$ .

In Figures 4.4, 4.5 and 4.6 we plot the regions in the parameter space  $(\alpha, q)$  corresponding to the different signs of  $(C, F)$ .

**Model III:**  $f(R) = R [\log(\alpha R)]^q - R$

This model may also belong to Class II of  $f(R)$  models presented in Section 2.4 if  $q > 0$  condition is satisfied and could be therefore cosmologically viable.

A vanishing curvature solution also appears in this model. The nontrivial one is given by

$$R_0 = \frac{1}{\alpha} \exp\left(\frac{2q}{D-2}\right). \quad (4.75)$$

Since  $R_0$  has to be negative,  $\alpha$  must be negative as well, accomplishing  $\alpha R_0 > 0$ . If  $q < 0$  is considered then, for  $D > 2$ , the expression (4.75) would imply  $\alpha R_0 < 1$  and then, from  $f(R)$  expression for this model, an inconsistency would appear since a negative number would be powered to a negative  $q$  value. Then  $q > 0$  is the only allowed interval for this parameter and therefore there exists a unique accessible region for parameters in this model:  $\alpha < 0$  and  $q > 0$ .

In Figures 4.7 and 4.8 we plot the regions in the parameter space  $(\alpha, q)$  corresponding to the different signs of  $(C, F)$ .

**Model IV:**  $f(R) = -\alpha m_1 \left(\frac{R}{\alpha}\right)^n \left[1 + \beta \left(\frac{R}{\alpha}\right)^n\right]^{-1}$

As was mentioned in Section 2.5, this model was originally proposed in [33] where it was considered to satisfy both cosmological and solar system tests without a cosmological constant. For this model,  $n = 1$  was considered for simplicity. Hence denoting  $f_R(R_0) \equiv \epsilon$  we get

$$m_1 = -\frac{(D - 2(1 + \epsilon))^2}{D^2 \epsilon} \quad (4.76)$$

and a relation between  $m_1$ ,  $D$  and  $\epsilon$  can be imposed. Therefore this model would only depend on two parameters  $\alpha$  and  $\beta$ . A vanishing curvature solution also appears in this

model and two nontrivial curvature solutions are given by:

$$R_0^\pm = \frac{\alpha}{2\beta(D-2)} \left[ D(m_1 - 2) + 4 \pm \sqrt{m_1} \sqrt{m_1 D^2 - 8D + 16} \right]. \quad (4.77)$$

The corresponding  $1 + f_R(R_0)$  values for (4.77) are

$$1 + f_R(R_0^\pm) = 1 - \frac{4(D-2)^2}{(\sqrt{m_1 D^2 - 8D + 16} \pm D\sqrt{m_1})^2} \quad (4.78)$$

where  $m_1 > 0$  and  $m_1 > (8D - 16)/D^2$  are required for  $R_0$  solutions to be real. Since  $1 + f_R(R_0) > 0$  is also a condition to be fulfilled, that means that  $\text{sign}(R_0^\pm) = \text{sign}(\alpha\beta)$ .

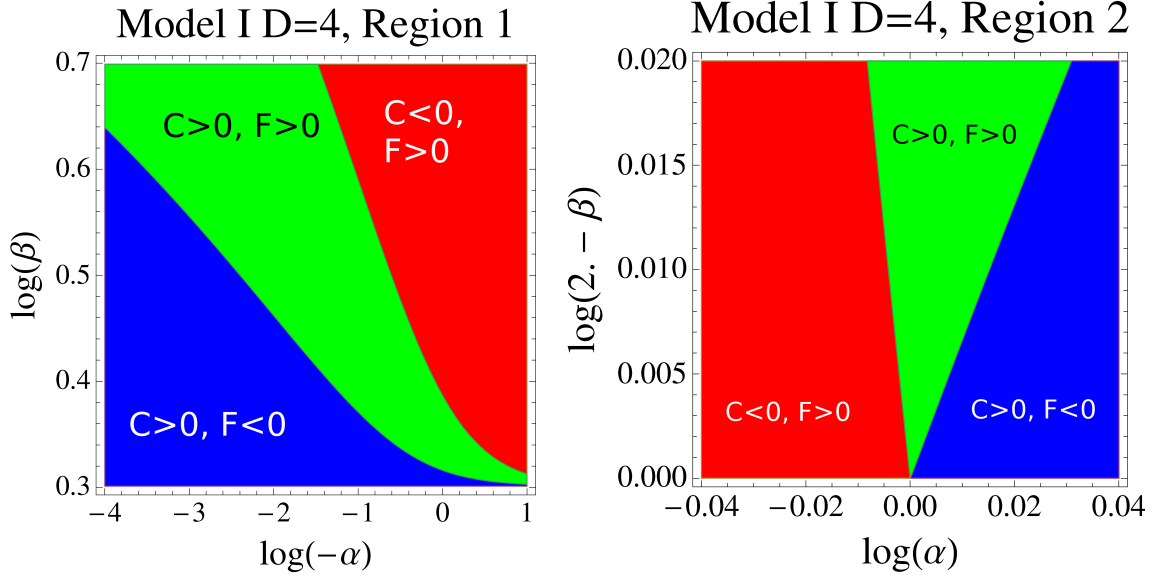
On the one hand, it can be shown that  $1 + f_R(R_0^-)$  is not positive for any allowed value of  $m_1$  and therefore this curvature solution  $R_0^-$  is excluded of our research. On the other hand,  $1 + f_R(R_0^+) > 0$  only requires  $m_1 > 0$  for dimension  $D \geq 4$  and therefore  $\epsilon < 0$  is required according to (4.76). Therefore only two accessible regions need to be studied: Region 1  $\{\alpha > 0, \beta < 0\}$  and Region 2,  $\{\alpha < 0, \beta > 0\}$ .

In Figures 4.9 and 4.10 we plot the thermodynamical regions in the parameter space  $(\alpha, \beta)$  for a chosen  $\epsilon = -10^{-6}$ . Note that  $1 + f_R(R_0^+)$  does not depend either on  $\alpha$  nor on  $\beta$  and that  $R_0^+$  only depends on the quotient  $\alpha/\beta$  for a fixed  $m_1$ .

## 4.6 Figures for thermodynamical regions

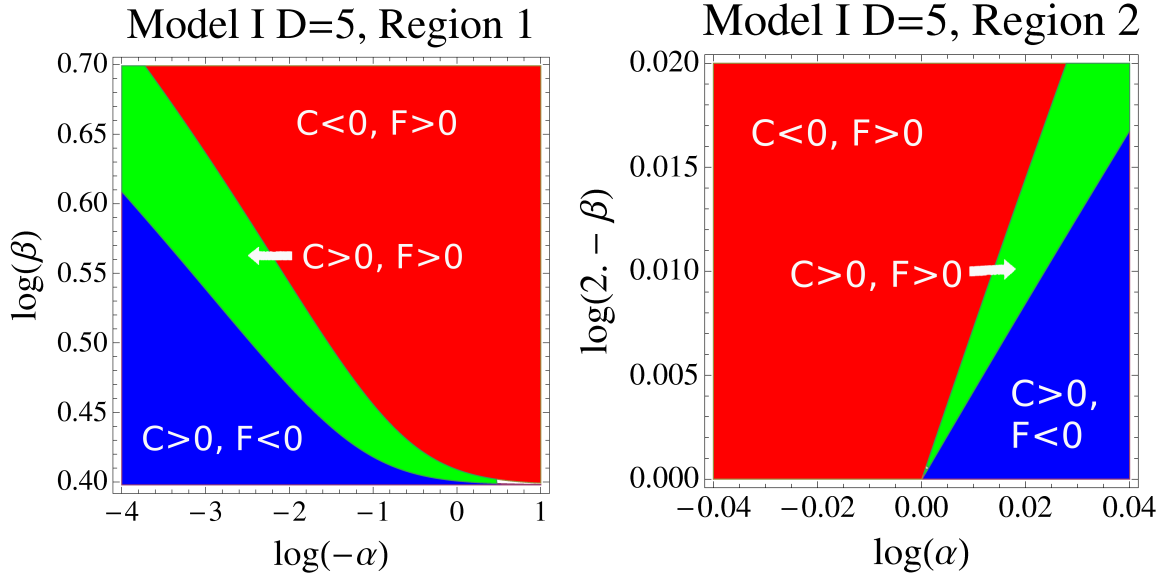
In the following pages we have plotted accessible thermodynamical regions for previously proposed  $f(R)$  models I-IV. Thermodynamical regions have been plotted using different colors: red:  $\{C < 0, F > 0\}$ , green:  $\{C > 0, F > 0\}$  and blue:  $\{C > 0, F < 0\}$ .

Parameter spaces have been chosen in order to show possible thermodynamical transitions between regions.



(a) Model I,  $D = 4$ , Region 1,  $\alpha < 0$ ,  $\beta > 2$ . (b) Model I,  $D = 4$ , Region 2,  $\alpha > 0$ ,  $\beta < 1$ .

Figure 4.1: Thermodynamical regions in the  $(\alpha, \beta)$  plane for Model I in  $D = 4$ . Region 1(left), Region 2 (right).



(a) Model I,  $D = 5$ , Region 1,  $\alpha < 0$ ,  $\beta > 2.5$ . (b) Model I,  $D = 5$ , Region 2,  $\alpha > 0$ ,  $\beta < 1$ .

Figure 4.2: Thermodynamical regions in the  $(\alpha, \beta)$  plane for Model I in  $D = 5$ . Region 1(left), Region 2 (right).

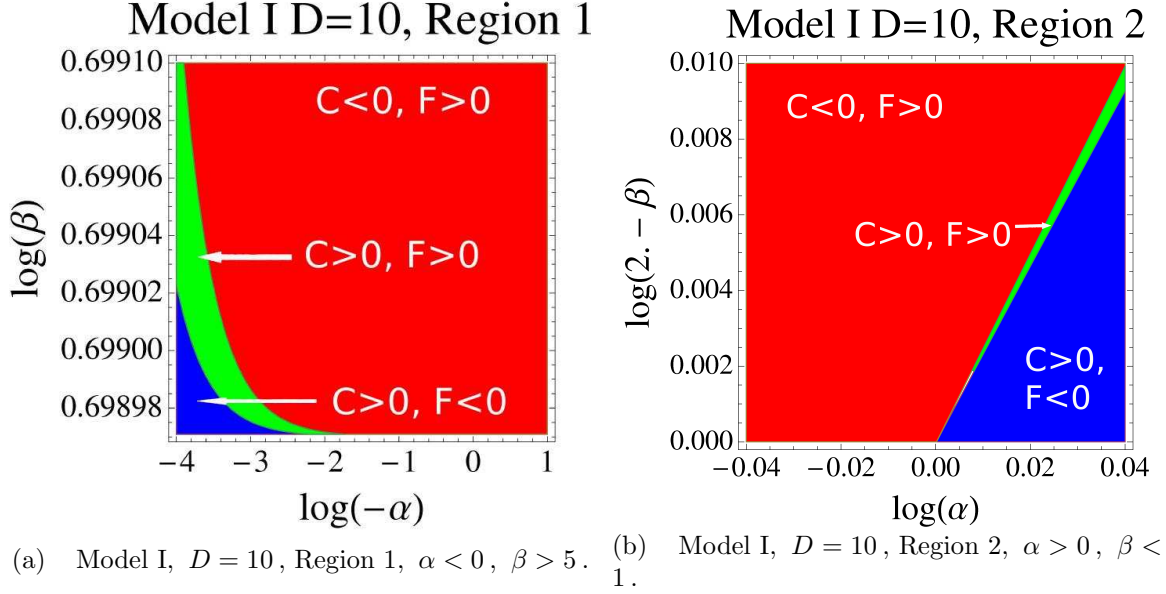


Figure 4.3: Thermodynamical regions in the  $(\alpha, \beta)$  plane for Model I in  $D=10$ . Region 1 (left), Region 2 (right).

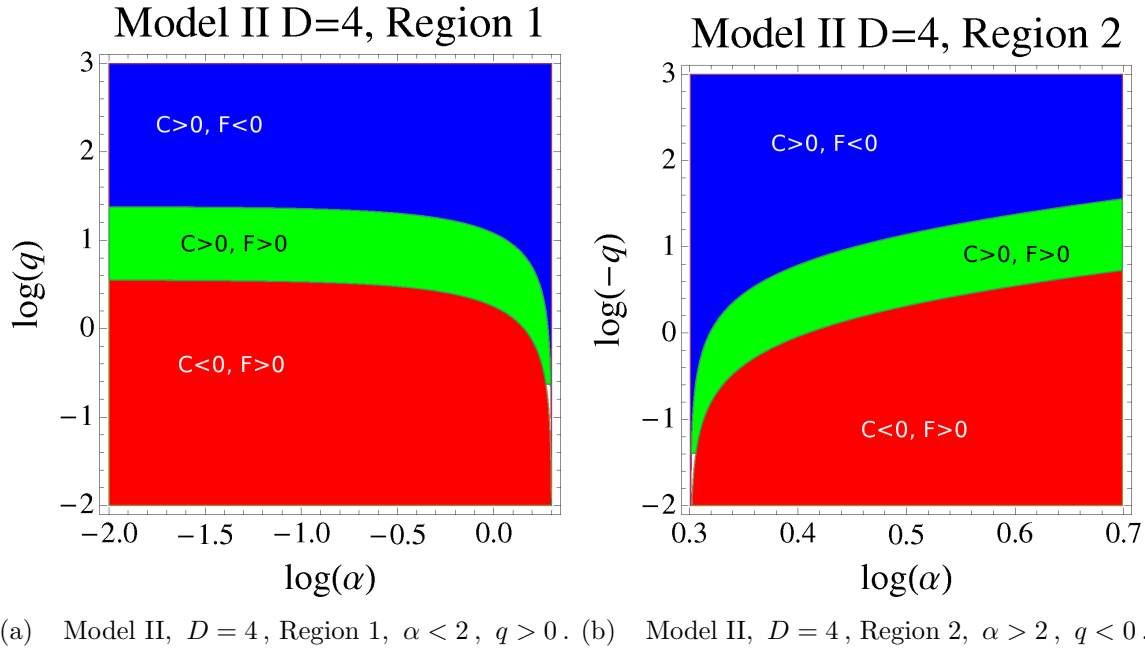
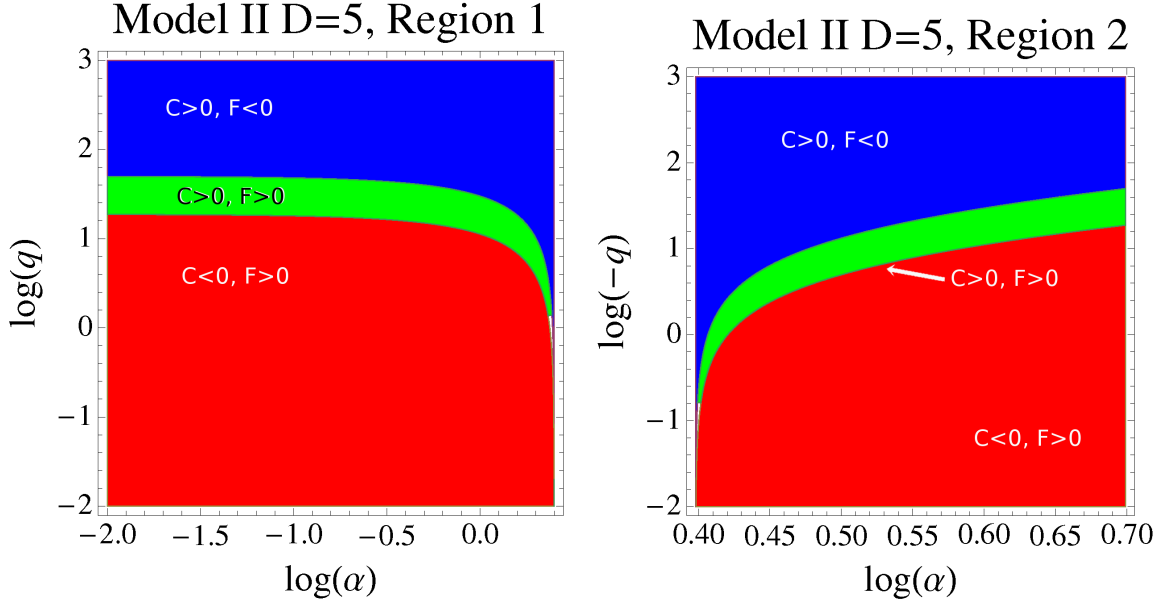
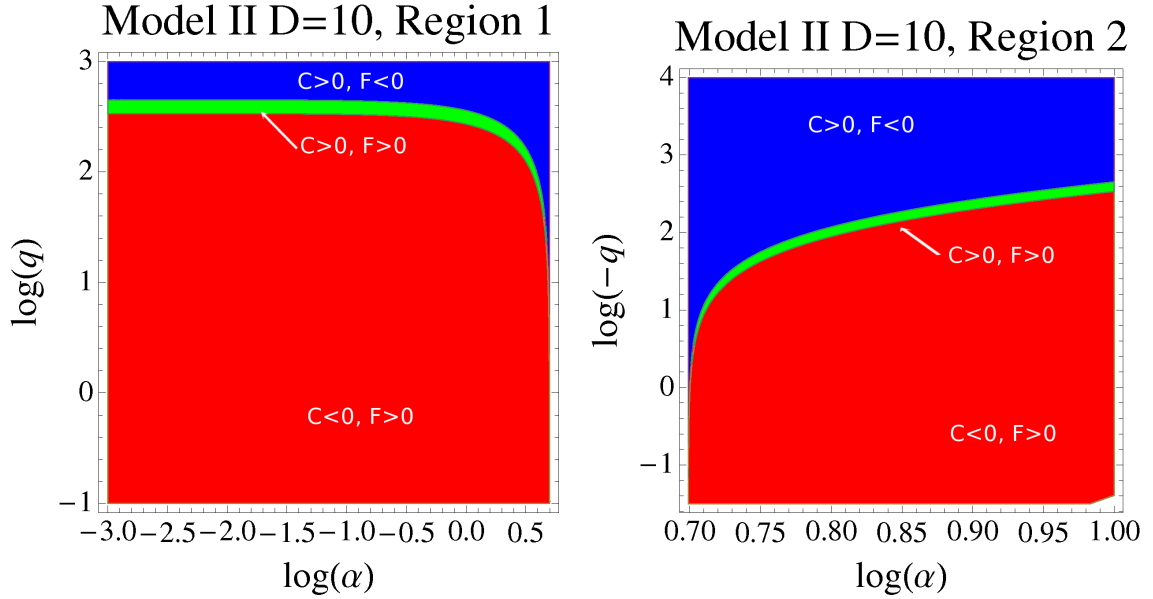


Figure 4.4: Thermodynamical regions in the  $(\alpha, q)$  plane for Model II in  $D=4$ . Region 1 (left), Region 2 (right).



(a) Model II,  $D = 5$ , Region 1,  $\alpha < 2.5$ ,  $q > 0$ . (b) Model II,  $D = 5$ , Region 2,  $\alpha > 2.5$ ,  $q < 0$ .

Figure 4.5: Thermodynamical regions in the  $(\alpha, q)$  plane for Model II in  $D = 5$ . Region 1(left), Region 2 (right).



(a) Model II,  $D = 10$ , Region 1,  $\alpha < 5$ ,  $q > 0$ . (b) Model II,  $D = 10$ , Region 2,  $\alpha > 5$ ,  $q < 0$ .

Figure 4.6: Thermodynamical regions in the  $(\alpha, q)$  plane for Model II in  $D = 10$ . Region 1(left), Region 2 (right).

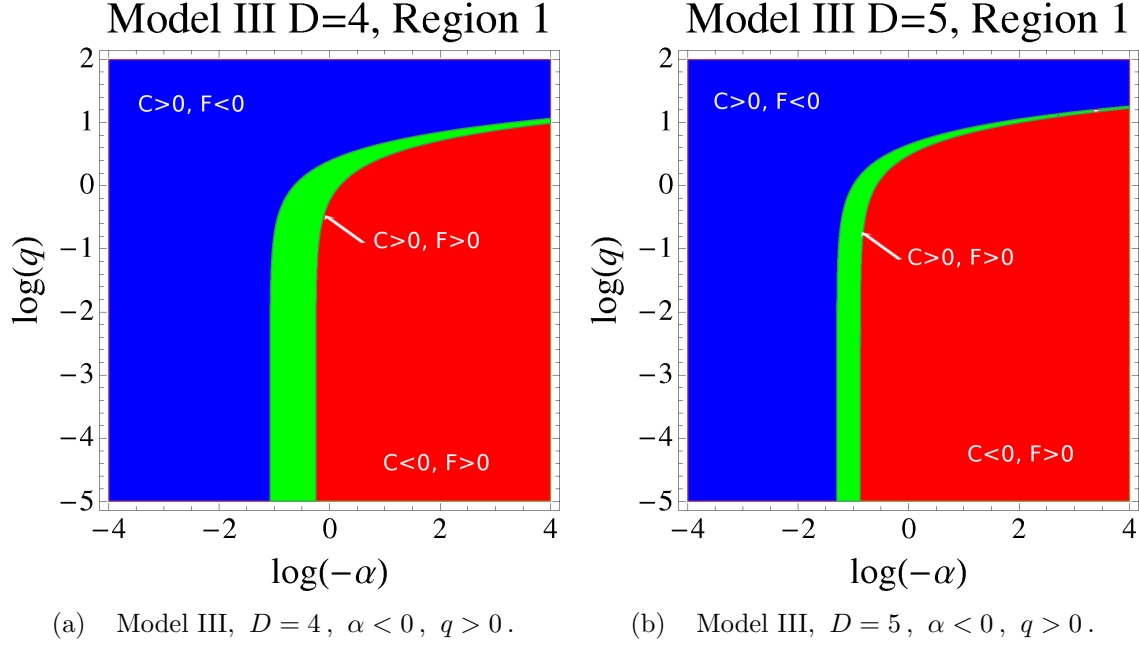


Figure 4.7: Thermodynamical regions in the  $(\alpha, q)$  plane for Model III in  $D = 4$  (left) and  $D = 5$  (right).

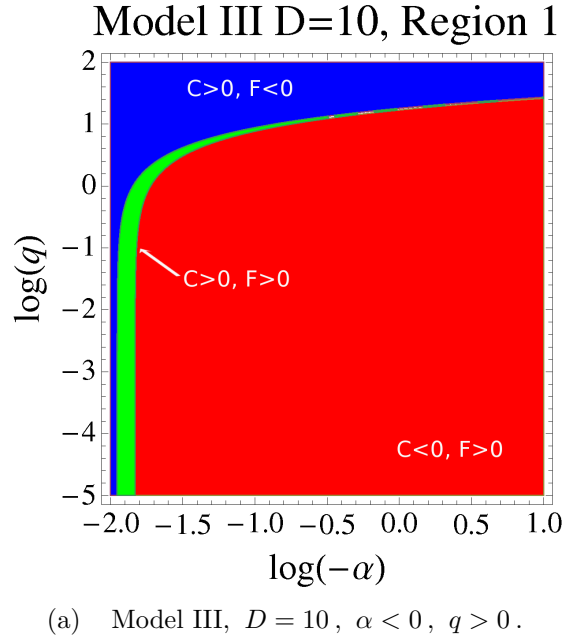


Figure 4.8: Thermodynamical regions in the  $(\alpha, q)$  plane for Model III in  $D = 10$ .

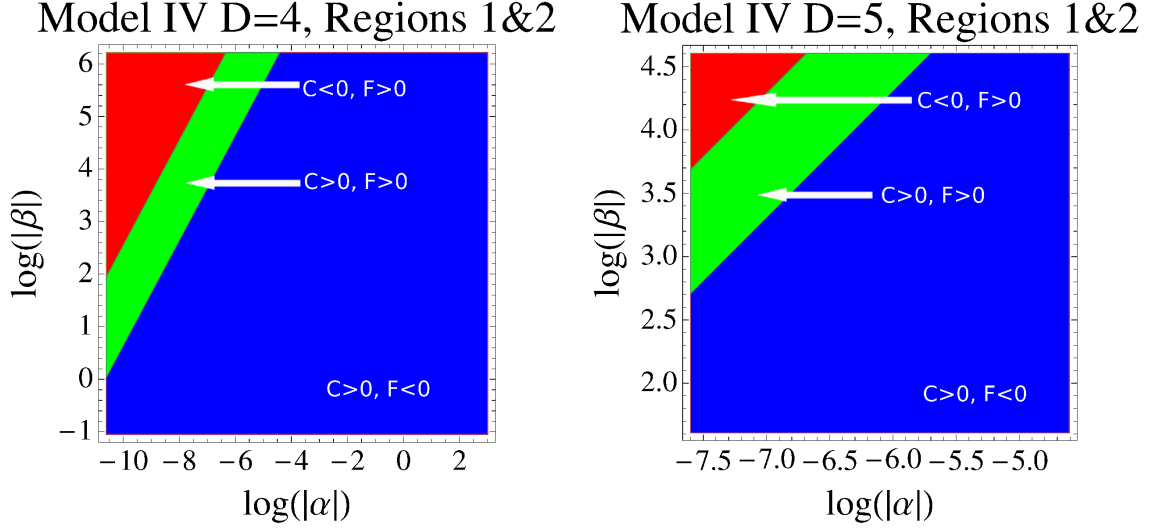


Figure 4.9: Thermodynamical regions in the  $(|\alpha|, |\beta|)$  plane for Model IV in  $D = 4$  (left) and  $D = 5$  (right).

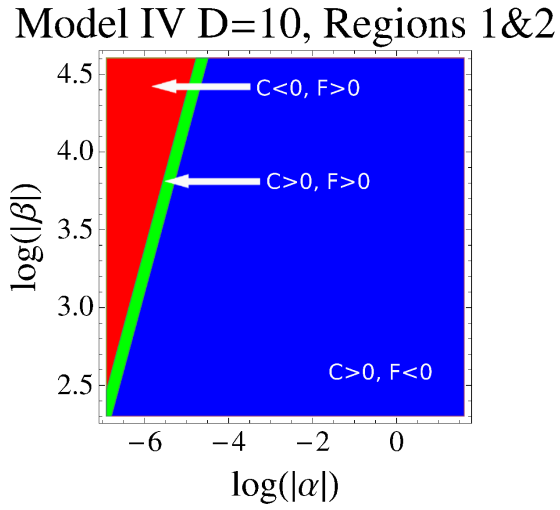


Figure 4.10: Thermodynamical regions in the  $(|\alpha|, |\beta|)$  plane for Model IV in  $D = 10$ .

## 4.7 Conclusions

In this chapter we have considered static spherically symmetric solutions in  $f(R)$  theories of gravity in arbitrary dimensions. Firstly, we have discussed the constant curvature case (including charged BH in 4 dimensions). Then, the general case, without imposing a priori the condition of constant curvature, has also been studied.

Another important result of this chapter was obtained by performing a perturbative analysis around the EH case, assuming regular  $f(R)$  functions. We have found explicit expressions up to second order for the metric coefficients. These coefficients only gave rise to constant curvature (Schwarzschild- $AdS$ ) solutions as in the EH case.

On the other hand, we have also calculated thermodynamical quantities for the  $AdS$  BHs and considered the issue of the stability of this kind of solutions. We have found that the condition for a  $f(R)$  theory of gravity to support this kind of BHs is given by  $R_0 + f(R_0) < 0$  where  $R_0$  is the constant curvature of the  $AdS$  space-time. This condition also implies that the effective Newton's constant is positive and that the graviton does not become a ghost. Consequently thermodynamical and cosmological viabilities of  $f(R)$  theories turned out to be related as we have shown.

Finally we have considered several explicit examples of  $f(R)$  functions and studied the parameter regions in which BHs in such theories are locally stable and globally preferred. It was found that the qualitative behaviour is the same as in standard EH gravity but the thermodynamical regions are modified depending on the parameters values in each case.

# Chapter 5

## Brane-skyrmions and the CMB cold spot

### 5.1 Introduction

Although the properties of BH solutions in BW models have been exhaustively studied in the literature, this is not the case of a different type of (topological) spherically symmetric solutions, the so-called brane-skyrmions. These configurations may appear in a natural way in a broad class of BW theories.

On the one hand, this type of textures can be understood as holes in the brane which make possible to pass through them along the extra-dimensional space. For such objects, cosmological involved scales and consequences have not been already studied in the literature. Therefore, the *leitmotiv* of this chapter will be precisely to study some astrophysical and cosmological effects that those configurations may have in the CMB data.

On the other hand, some striking features have recently been discovered [101] in the CMB data such as the presence of an anomalous cold spot (CS) in the WMAP temperature maps. This CS has been interpreted in different manners by making use of different physical mechanisms. Among them, one of the most exciting from the physical point of view could justify the appearance of cold spots as the result of a collapsing texture, perhaps coming from some early universe Grand Unified Theory (GUT) phase transition.

In this chapter we shall propose an alternative explanation to those in the existing literature: it will be shown that the brane-skyrmions provide a natural scenario to reproduce the CS features and that typical involved scales, needed for the proposed brane-skyrmions to describe correctly the observed CS, can be as low as the electroweak scale.

The present chapter is thus organized as follows: brane-skyrmions in brane-worlds are presented in Section 5.2 and then we shall determine the possible distortion in the fractional profile of temperatures generated by the presence of these objects. Next, in Section 5.3 we shall present the most recent results that claim the existence of a CS in the CMB data. Different possible explanations that have been previously proposed in the literature will be summarized in Section 5.4.

The subsequent physical interpretation of the calculations performed in Section 5.2 will be presented in Section 5.5. In this section, we will also show that those results are in agreement with the CMB data and with other theoretical proposals. To conclude the chapter, possible future detection of these brane-skyrmions and some conclusions will be studied in Section 5.6.

The results of this chapter were originally published in [102].

## 5.2 Spherically symmetric brane-skyrmions

In this section we are going to generalize the results presented in Section 1.8 for static brane-skyrmions. Here we shall allow time dependence for these objects but spherical symmetry will be preserved. Starting from the physical branon fields equations provided in equation (1.65), one may define the brane-skyrmion spherical coordinates with winding number  $n_W$  by the nontrivial mapping  $\pi^\alpha : S^3 \longrightarrow S^3$  as follows:

$$\phi_K = \phi, \quad \theta_K = \theta, \quad \chi_K = F(t, r) \quad (5.1)$$

with boundary conditions satisfying the requirement

$$F(t, \infty) - F(t, 0) = n_W \pi. \quad (5.2)$$

This map is usually referred to as the *hedgehog* ansatz. With these coordinates (5.1) introduced in the metric (1.58), the Nambu-Goto action given in expression (1.53) may be rewritten in terms of the previous coordinates as follows:

$$S_{NG} = -f^4 \int d^4x \sin\theta \left( 2 \frac{v^2}{f^4} \sin(F) \cos(F) \right) \left[ 1 - \frac{v^2}{f^4} (\dot{F}^2 - F'^2) \right]^{1/2}. \quad (5.3)$$

Varying this action with respect to the function  $F(t, r)$  the equation of motion for the skyrmion profile is obtained and it becomes:

$$\sin(2F) - 2rF' + \left( r^2 + \frac{v^2}{f^4} \sin^2 F \right) \frac{\ddot{F} - F'' + \frac{v^2}{f^4} (\ddot{F}F'^2 + F''\dot{F}^2 - 2F'\dot{F}\dot{F}')}{1 - \frac{v^2}{f^4} (\dot{F}^2 - F'^2)} = 0 \quad (5.4)$$

where dot and prime denote – throughout this section – derivatives with respect to  $t$  and  $r$  respectively.

In this chapter we are interested in the potential cosmological effects due to the presence of a brane-skyrmion within our Hubble radius. For that purpose, gravitational field perturbations will be computed at large distances compared to the size of the extra dimensions, i.e., we are interested in the region

$$r^2 \gg R_B^2 = \frac{v^2}{f^4}. \quad (5.5)$$

Notice that in this region, gravity behaves essentially as in four dimensional space-time and standard GR can be used in the calculations. Notice also that in order to simplify these calculations, the effects due to the universe expansion will be ignored or at least assumed negligible. This assumption is fully justified provided  $r \ll H_0^{-1}$ . In such a case the unperturbed (ignoring the presence of the defect) background metric can be taken as Minkowski, i.e.,  $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$  in expression (1.58). From assumptions in (5.5), the equation of motion (5.4) reduces to:

$$r^2 \left( \ddot{F}_0 - F_0'' \right) + \sin(2F_0) - 2rF_0' = 0 \quad (5.6)$$

which is equivalent to expression (3) in reference [103]. Notice that this is an expected result since, as shown in [104, 105], at large distances, i.e. except in the microscopic unwinding regions, the dynamical evolution of the fields is completely independent from the symmetry breaking mechanism, it simply depends on the geometry of the coset manifold  $K$ . On small scales [104, 105] it is possible that higher-derivative terms could affect the dynamics and even stabilize the textures, this is also the case of brane-skyrmions [49], although generically they could unwind by means of quantum-mechanical effects.

The equation (5.6) admits an exact solution with winding number  $n_W$  equals to unity of the following form:

$$F_0(t, r) = 2 \arctan \left( -\frac{r}{t} \right) \quad (5.7)$$

with  $t < 0$  since as explained below, we will be interested in photons passing the texture before it collapses.

Our approximated equation (5.6) is consistent with the complete equation (5.4) for the above solution (5.7) since the second term in the numerator in equation (5.4) vanishes for  $F \equiv F_0$  and

$$\dot{F}_0^2 - F_0'^2 = 4 \frac{r^2 - t^2}{(r^2 + t^2)^2} \quad ; \quad \sin^2 F_0 = 4 \frac{r^2 t^2}{(r^2 + t^2)^2} \quad (5.8)$$

so that in the considered regime, the neglected terms in equation (5.4) are indeed irrelevant for all  $r$  and  $t$  values.

Now that  $F_0(t, r)$  has been determined, we may calculate the energy-momentum tensor components also in this region from the Nambu-Goto action (1.53) as:

$$T^{\mu\nu} = 2|\tilde{g}|^{-1/2} \frac{\delta S_{NG}}{\delta \tilde{g}_{\mu\nu}} \quad (5.9)$$

where note that  $\tilde{g}$  is the involved metric to determine this tensor. In spherical coordinates they become

$$\begin{aligned} T_{00} &= -\frac{2v^2(r^2 + 3t^2)}{(t^2 + r^2)^2} ; \quad T_{rr} = \frac{2v^2(t^2 - r^2)}{(t^2 + r^2)^2} \\ T_{0r} &= \frac{4v^2rt}{(t^2 + r^2)^2} ; \quad T_{\theta\theta} = \frac{2v^2r^2(t^2 - r^2)}{(t^2 + r^2)^2} \\ T_{\phi\phi} &= \sin^2\theta T_{\theta\theta} \end{aligned} \quad (5.10)$$

and note that  $\nabla_\mu T^\mu_\nu$  identically vanishes.

We shall now determine the background metric  $\tilde{g}_{\mu\nu}$  in the  $r \gg R_B$  region and in the presence of the brane-skyrmion as a small perturbation on the Minkowski metric, i.e.  $\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ .

Thus, for the scalar perturbation of the Minkowski space-time in the longitudinal gauge, the line element will adopt the form, straightforwardly obtained from equation (3.18) in Chapter 3 if  $a(\tau) \equiv 1$ , that follows:

$$ds^2 = (1 + 2\Phi)dt^2 - (1 - 2\Psi)(dr^2 + r^2 d\Omega_2^2) \quad (5.11)$$

and thus the perturbed Einstein tensor components in cartesian coordinates are the following (see [106]):

$$\begin{aligned} \delta G_0^0 &= 2\nabla^2\Psi \\ \delta G_i^j &= -[2\ddot{\Psi} + \nabla^2(\Phi - \Psi)]\delta_i^j + \partial_i\partial^j(\Phi - \Psi) \\ \delta G_i^0 &= 2\partial_i\dot{\Psi} \end{aligned} \quad (5.12)$$

with  $i, j = 1, 2, 3$  and  $\nabla^2 \equiv \sum_{i=1}^3 \partial_i\partial^i$ . Using Einstein's equations  $\delta G^\mu_\nu = -8\pi G T^\mu_\nu$  we determine that the potentials  $\Phi$  and  $\Psi$  are

$$\Psi \equiv \Phi = 4\pi G v^2 \log\left(\frac{r^2 + t^2}{t^2}\right). \quad (5.13)$$

The physical metric on which photons propagate is not the  $\tilde{g}_{\mu\nu}$  we have just calculated, but the induced metric (1.58). However, using the solution in (5.7), we find that the contribution from branons fields is  $\mathcal{O}(R_B^2/r^2)$ , i.e. negligible in the region we are interested in, so that  $g_{\mu\nu} \simeq \tilde{g}_{\mu\nu}$ .

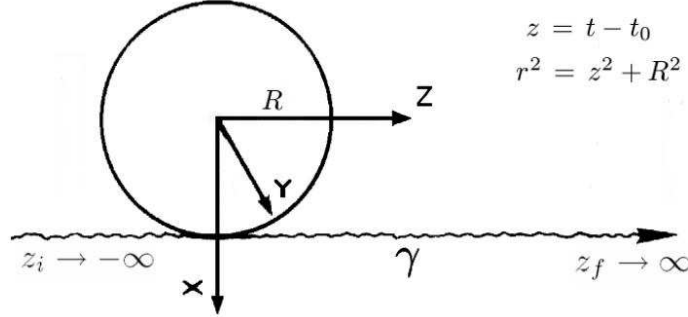


Figure 5.1: Schematic representation of photon trajectory passing near the brane-skyrmion as originally presented in [103].  $R$  is the impact parameter (the radius in the plotted circle) and the photon trajectory through the bottom horizontal line is chosen along the  $Z$  axis.

Photons propagating on the perturbed metric will suffer red (blue)-shift due to the Sachs-Wolfe (SW) effect [106]. The full expression for the temperature fluctuation is given by:

$$\left(\frac{\Delta T}{T}\right)_{\text{SW}} = -[\Phi]_{\tau_i}^{\tau_f} + \int_{\tau_i}^{\tau_f} (\dot{\Psi} + \dot{\Phi}) d\tau = -[\Phi]_{\tau_i}^{\tau_f} + \int_{\tau_i}^{\tau_f} 2\dot{\Phi} d\tau \quad (5.14)$$

where we have considered local and integrated SW effects and neglected the Doppler contribution.  $\tau_i$  and  $\tau_f$  are the initial and final times respectively required to study temperature fluctuation in that interval. Substituting expression (5.13) in the previous one and using that

$$r^2 = z^2 + R^2 ; \quad z = t - t_0 \quad (5.15)$$

as may be seen at the schematic representation in Figure 5.1, the fractional distortion we obtain is:

$$\left(\frac{\Delta T}{T}\right)_{\text{SW}} = 8\pi G v^2 \left( \frac{t_0}{\sqrt{2R^2 + t_0^2}} \arctan \left( \frac{t_0 + 2z}{\sqrt{2R^2 + t_0^2}} \right) - \log|t_0 + z| \right)_{z_i}^{z_f} \quad (5.16)$$

where  $R$  is the impact parameter and  $t_0$  is the time at which the photon passes the texture position at  $z = 0$ . In the limit where  $z_i \rightarrow -\infty$  and  $z_f \rightarrow \infty$  the result is

$$\left(\frac{\Delta T}{T}\right)_{\text{SW}} = \epsilon \frac{t_0}{\sqrt{2R^2 + t_0^2}} \quad (5.17)$$

with  $\epsilon \equiv 8\pi^2 G v^2$ . As shall be seen in the next section, this temperature profile is the one which fits the apparently observed anomaly in the CMB data.

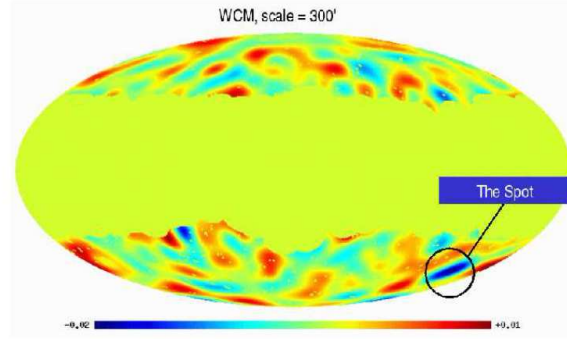


Figure 5.2: The combined and foreground cleaned Q-V-W WMAP map after convolution with the SMHW at scale  $R_9$  as shown in [115]. The CS position is marked and seen at the bottom left part of the plot.

### 5.3 Cold spot in WMAP data

One of the most important pieces of information about the history and nature of our universe comes from the Cosmic Microwave Background (CMB). Measurements of the CMB temperature anisotropies obtained by WMAP [11, 107] have been thoroughly studied in recent years. Such anisotropies have been found to be Gaussian as expected in many standard cosmological scenarios corresponding to density fluctuations of one part in a hundred thousand in the early universe. However, by means of a wavelet analysis, an anomalous CS, apparently inconsistent with homogeneous Gaussian fluctuations, was found in [101, 108] in the southern hemisphere centered at the position  $b = -57^\circ$ ,  $l = 209^\circ$  in galactic coordinates.<sup>1</sup> The characteristic scale in the sky of the CS is about  $5^\circ$ .

The CS position in usual WMAP plots is shown in Figure 5.2. The existence of this CS has been claimed to be confirmed more recently in reference [109]. However, in recent references [110] it has been argued that there is no compelling evidence for deviations from the  $\Lambda$ CDM model in the WMAP data. In particular, it is claimed that the evidence that the CS is statistically anomalous is not robust.

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<sup>1</sup>where  $b$  is the galactic latitude measured from the plane of the galaxy to the object using the Sun as vertex. The galactic longitude is referred to as  $l$  and it is measured in the plane of the galaxy using an axis pointing from the Sun to the galactic center.

Concerning the origin of the CS, different explanations have been proposed in the literature: in references [111, 112, 113] the CS was tried to be explained by voids in the matter distribution whereas the reference [114] dealt with possible string theory bubble origin. Other authors proposed in [115] the Sunyaev-Zeldovich (SZ) effect as possible explanation showing that the flat frequency dependence of the CS is incompatible with being caused by a SZ signal alone. However, a combination of CMB plus SZ effect may explain the spot and could have a sufficiently flat frequency spectrum.

## 5.4 Cold spot as a cosmic texture

A very interesting possibility about the CS origin has recently been proposed in [116]. According to it, some theories of high energy physics predict the formation of various types of topological defects, including cosmic textures [104] which would generate hot and cold spots in the CMB [103]. These textures would be the remnants of a symmetry breaking phase transition that took place in the early universe.

In order to produce textures, the cosmic phase transition must be related to a global symmetry breaking pattern from one group  $G$  to a subgroup  $H$  so that the coset space  $K = G/H$  has a nontrivial third homotopy group. A typical example is  $K = SU(N)$ , which has associated  $\pi_3(K) = \mathbb{Z}$  for  $N \geq 2$ . Notice that, as usual, in order to get a texture formed in the transition, the symmetry breaking must correspond to a global symmetry since if it were local it could be gauged away.

Textures can be understood as localized wrapped field configurations which collapse and unwind on progressively larger scales. These textures can produce a concentration of energy which gives rise to a time dependent gravitational potential. CMB photons traversing the texture region will suffer a red (blue) shift producing a cold (hot) spot in the CMB maps. In reference [116] the authors consider a  $SU(2)$  non-linear sigma model to build up a model of texture that could explain the observed CS. They simulate the unwinding texture by using a spherically symmetric scaling solution and they find a fractional temperature distortion given by:

$$\frac{\Delta T}{T}(\theta) = \pm \epsilon \frac{1}{\sqrt{1 + 4 \left( \frac{\theta}{\theta_c} \right)^2}} \quad (5.18)$$

where  $\theta$  is the angle from the center,  $\epsilon$  is a measure of the amplitude and  $\theta_c$  is the scale parameter that depends on the time at which the texture unwinds. The best fit of the CS is found for  $\epsilon = 7.7 \times 10^{-5}$  and  $\theta_c = 5.1^\circ$ . Furthermore, the parameter  $\epsilon$  is given by  $\epsilon \equiv 8\pi^2 G \Phi_0$ , where  $\Phi_0$  is the fundamental symmetry breaking scale which is then set to be  $\Phi_0 \simeq 8.7 \times 10^{15}$  GeV. This scale is nicely close to the GUT scale thus making the results given in [116] extremely interesting.

Nonetheless, it is important to stress that textures require having a global symmetry breaking but what one finds typically in GUTs is a local symmetry breaking producing the Higgs mechanism and then destroying the topological meaning of texture or any other possible defect appearing in the cosmic transition.

## 5.5 Physical interpretation of the results and involved scales

As has been shown in Sections 5.2 and 5.4, the results for the fractional temperature distortion provoked by the CS given by expression (5.18) coincides with the fractional distortion caused by a brane-skyrmion given by expression (5.17) provided that:

$$\Phi_0 = v \ ; \ 2 \left( \frac{\theta}{\theta_c} \right) = \frac{R}{t_0} \quad (5.19)$$

where  $\theta_c$  was defined in reference [116].

Let us now estimate the required scales in order for this kind of brane-skyrmion to be able to explain the observed CS. For the minimal model supporting brane-skyrmions with three extra dimensions  $\delta = 3$ , the relation (1.50) approximately becomes

$$M_P^2 \simeq R_B^3 M_D^5 \quad (5.20)$$

since gravitation is embedded in  $D = 7$  dimensions and therefore  $R_B^3$  is the characteristic volume for the compactified extra dimensions. Therefore

$$R_B \simeq \left( \frac{M_P^2}{M_D^5} \right)^{1/3} \quad (5.21)$$

and consequently

$$v^2 \equiv f^4 R_B^2 \simeq f^4 \left( \frac{M_P^2}{M_D^5} \right)^{2/3} . \quad (5.22)$$

If the parameter  $\epsilon$  needs to be fixed around  $7.7 \cdot 10^{-5}$ , then the required  $v$  would be

$$v \simeq 1.2 \cdot 10^{16} \text{ GeV} \quad (5.23)$$

which in fact can be achieved with  $M_D \sim f \sim \text{TeV}$ . Notice that for this parameter range, the radius of the extra dimension is around

$$R_B \sim 10^{-8} \text{ m}, \quad (5.24)$$

i.e., previous approximations in expression (5.5) are totally justified.

In addition to the previous estimations, we have also checked that the possible effects coming from a nonvanishing branon mass are suppressed by  $\mathcal{O}(M^2/v^2)$ . Therefore for mass values also around  $M \sim \text{TeV}$ , which are typical of branon as dark matter candidate, see [25], such effects are negligible.

In other words, brane-skyrmions provide an accurate description for the CS without any need of introducing very high energy (GUTs) scales since the correct temperature fluctuation amplitude can be obtained with natural values around the electroweak scale.

### 5.5.1 Brane-skyrmions abundance

The abundance of brane-skyrmions in this model is given by its low-energy (large distance) dynamics as described by a non-linear sigma model as that in [116]. This is nothing but the well-known fact that, except in the microscopic unwinding region, the field evolution only depends on the geometry of the coset space  $K$ , but not on the details of the symmetry breaking mechanism. As was shown in [105] in a simple model with a potential term, the final abundance of defects and other properties of the pattern of density perturbations are expected to be not very sensitive to the short distance physics, once the texture unwinds, making this kind of theories highly predictive. For that reason, we expect that provided the same kind of initial conditions are imposed in both models, the predicted abundance of hot and cold spots agree with that obtained from simulations in [116]. Such simulations show that the number of unwinding textures per comoving volume and conformal time  $\tau$  can be estimated as

$$\frac{dn}{d\tau} = \frac{\nu}{\tau^4} \quad (5.25)$$

with  $\nu \simeq 2$ . This allows to estimate the number of hot and cold spots in a given angular radius interval. As commented above, this is a quite robust result at late times with little effects for short distance dynamics. In the case of brane-skyrmions, the short distance effects will be embodied in the higher-derivative terms appearing in the expansion of the Nambu-Goto action or even in possible induced curvature terms generated by quantum effects [49]. Provided that such terms do not stabilize the brane-skyrmions, we expect that such an abundance could be directly applied in our case.

## 5.6 Future prospects and conclusions

In this section we dedicate some lines to mention how both present and future experiments may confirm the validity of the BW models. Then we shall sketch the most important conclusions and consequences of the brane-skyrmion model presented in this chapter as a viable explanation for the observed CS.

On the one hand, the fact that the fundamental scales of the theory are of the order of the TeV opens the possibility to test this explanation with collider experiments through the production of real or virtual branons and KK-gravitons. The expected signatures of the model [117] from the production of KK-gravitons come fundamentally from the single photon channel studied by LEP, which restrict  $M_D > 1.2$  TeV at 95% of confidence level.

On the other hand, the LHC accelerator will be able to test the model up to  $M_D = 3.7$  TeV, analysing single photon and monojet production to establish the number of extra dimensions as was explained in Section 1.10.

Finally, there also exists the possibility to find signatures of the model at low energies associated with branon phenomenology. This case is more interesting from the cosmological point of view, since branons can constitute the nonbaryonic dark matter abundance as typical WIMPs [25, 118]. Present constraints coming from the single photon analysis realized by L3 (LEP) imply  $f > 122$  GeV (at 95 % C.L.) [119, 120] and the LHC will be able to check this model up to  $f = 1080$  GeV through monojet production [118]. The idea to test the physics associated with the CS with the next generation of colliders at the TeV scale is a very intriguing and distinctive property of this texture.

## Chapter 6

# Conclusions and prospects

In this thesis we have studied some cosmological and astrophysical consequences in two different types of modified gravity theories:  $f(R)$  theories and brane-world extra dimensions theories.

First, we have shown some relevant results for  $f(R)$  modified gravities. Thus, we have studied how some features of general relativity can be mimicked by  $f(R)$  models if certain conditions are imposed on those models. We have also proved that there exists a class of  $f(R)$  models which can indeed reproduce the present cosmological evolution –from matter-radiation equality till today – as described by general relativity within the standard cosmological concordance model. These  $f(R)$  models allow to remove the cosmological constant term and therefore the observed acceleration is understood to have a purely geometrical interpretation with no dark energy origin. Furthermore, initial conditions may be imposed on such a class of theories in order to recover a null scalar curvature solution for a vacuum scenario. Unfortunately such functions are not cosmologically viable and should be considered as effective models to reproduce  $\Lambda$ CDM evolution.

Also, it has been proved that any perfect fluid parameterized by a constant equation of state can be reproduced by the presence of  $f(R)$  terms. Thus, these functions could provide a mechanism to reproduce the cosmological behaviour of dark energy type fluids. It therefore seems worthwhile to try to search for a unique phenomenological  $f(R)$  theory able to explain the main features in the cosmological evolution, from inflation till late time observed acceleration.

Next, we have considered the modification introduced by the new  $f(R)$  terms in the evolution of cosmological perturbations. Here, an analysis concerning first order cosmological scalar perturbations has been performed. We have presented a general method in order to obtain the evolution equation of the matter density contrast for arbitrary  $f(R)$  theories. That procedure was proven to be valid regardless of the chosen  $f(R)$  model or the size of

the involved scales. An important by-product of the calculations is the conclusion that the usual approximations made in the literature for sub-Hubble modes are not always valid. In fact, the required hypotheses to get the so-called quasi-static approximation have been established here. For those sub-Hubble modes, it was found that only for  $f(R)$  models satisfying the local gravity constraints, the evolution of perturbations is indistinguishable from that obtained using the quasi-static approximation but perfectly distinguishable from the evolution obtained by the  $\Lambda$ CDM model.

Now that those theoretical calculations were made, the correct evolution for matter perturbations in sub-Hubble modes for  $f(R)$  models is obtained. Therefore some robustness tests may be used to compare experimental data with expected theoretical results. Thus both the validity and certain constraints on those  $f(R)$  functions may be established. This procedure shows how our theoretical results may be used in order to confront the predictions made by a given  $f(R)$  theory with large scale structure observations.

As a final subject within our  $f(R)$  models research, we dealt with some features of black holes in arbitrary dimensions. First, the scalar constant curvature case for static and spherically symmetric solutions in vacuum was studied. It was determined that the only possible solution for the modified Einstein equations in this case was the Schwarzschild-(anti)-de Sitter solution. As a complementary result to the previous research, we studied the static and spherically symmetric case without imposing the constant curvature condition. To do so, a perturbative approach was performed around the standard Schwarzschild-anti-de Sitter solution of general relativity. It was found that, up to second order in perturbations, the only solution was the generalized Schwarzschild-anti-de Sitter solution with modified coefficients in terms of  $f(R)$  and its derivatives evaluated at the background curvature.

This part of the thesis was finished by studying the thermodynamic properties of Schwarzschild-anti-de Sitter black holes in  $f(R)$  theories. Thus accessible thermodynamical regions for several  $f(R)$  models were studied in detail. In that realm, a very interesting property, which encourages further investigation, was found: it turned out that thermodynamical viability of  $f(R)$  theories in constant curvature solutions was embodied by the condition  $1 + df(R)/dR > 0$ . This condition was also required to ensure gravitational viability for  $f(R)$  models. Consequently two different aspects of the viability of  $f(R)$  theories have been brought together.

The last part of the thesis was devoted to a study of some consequences of the existence of topologically nontrivial brane configurations, the so-called brane-skyrmions, in brane-world theories. In particular we have considered the effects of brane-skyrmions on the temperature fluctuations of the CMB and their potential connection with the so-called CMB cold spot. We have shown that brane-world theories can naturally accommodate this striking cosmological feature by invoking these nontrivial topological configurations. A different type of textures related to Grand Unified Theories has also been considered in the literature, but the model presented here shows that this type of defect could be

explained from the electroweak scale physics.

To conclude, the results included in this thesis have shown that modified gravity theories remain compelling candidates to describe the properties of the gravitational interaction on very large scales. Both the validity and viability of these theories have still to be subjected to many theoretical and experimental tests. To do so, the possibility of reproducing standard cosmological results, first order perturbations, black holes and features on the CMB temperature maps were considered as interesting aspects which could shed some light on the fundamental properties of the gravitational interaction.



## Appendix A

### Coefficients in $f(R)$ cosmological perturbations

In this appendix we include the coefficients for the series expansions given in equation (3.50) in Chapter 3. All the  $\alpha$ 's terms (coming from EH-part) and the first four  $\beta$ 's terms (coming from  $f$ -part) for each  $\delta$  derivative, i.e.  $\delta^{iv}, \dots, \delta, ,$  in equation (3.50) have been included once the condition  $|f_R| \ll 1$  has been imposed to simplify their expressions. For sub-Hubble scales the excluded  $\beta$ 's terms are negligible with respect to the written ones.

#### A.1 Appendix I: $\alpha$ 's and $\beta$ 's coefficients

Coefficients for  $\delta^{iv}$  term:

$$\begin{aligned}\beta_{4,f}^{(1)} &\simeq 8f_R^4(1+f_R)^6f_1^4\epsilon^2 \\ \beta_{4,f}^{(2)} &\simeq 72f_R^3f_1^3\epsilon^4(-2+\kappa_2) \\ \beta_{4,f}^{(3)} &\simeq 216f_R^2f_1^2\epsilon^6(-2+\kappa_2)^2 \\ \beta_{4,f}^{(4)} &\simeq 216f_Rf_1\epsilon^8(-2+\kappa_2)^3.\end{aligned}\tag{A.1}$$

Coefficients for  $\delta'''$  term:

$$\begin{aligned}\beta_{3,f}^{(1)} &\simeq 8f_R^4(1+f_R)^5f_1^4\mathcal{H}\epsilon^2[3+f_R(3+f_1)] \\ \beta_{3,f}^{(2)} &\simeq 6f_R^3f_1^2\mathcal{H}\epsilon^4\{8f_2(-2+\kappa_2)+4f_1[12\kappa_1+9\kappa_2-2(9+\kappa_3)]\} \\ \beta_{3,f}^{(3)} &\simeq -72f_R^2f_1\mathcal{H}\epsilon^6(-2+\kappa_2)[-4f_2(-2+\kappa_2)+f_1(19-23\kappa_1-10\kappa_2+4\kappa_3)] \\ \beta_{3,f}^{(4)} &\simeq -216f_R\mathcal{H}\epsilon^8(-2+\kappa_2)^2[-2f_2(-2+\kappa_2)+f_1(7-11\kappa_1-4\kappa_2+2\kappa_3)].\end{aligned}\tag{A.2}$$

Coefficients for  $\delta''$  term:

$$\begin{aligned}\alpha_{2,EH}^{(1)} &= 432(1+f_R)^{10}\mathcal{H}^2\epsilon^8(-1+\kappa_1)(-2+\kappa_2)^3 \\ \alpha_{2,EH}^{(2)} &= 1296(1+f_R)^{10}\mathcal{H}^2\epsilon^{10}(-1+\kappa_1)^2(-2+\kappa_2)^3 \\ \alpha_{2,EH}^{(3)} &= 3888(1+f_R)^{10}\mathcal{H}^2\epsilon^{12}(-1+\kappa_1)^2(-2+\kappa_2)^3 \\ \beta_{2,f}^{(1)} &\simeq 8f_R^4(1+f_R)^6f_1^4\mathcal{H}^2 \\ \beta_{2,f}^{(2)} &\simeq 88f_R^3f_1^3\mathcal{H}^2\epsilon^2(-2+\kappa_2) \\ \beta_{2,f}^{(3)} &\simeq 24f_R^2f_1^2\mathcal{H}^2\epsilon^4(-2+\kappa_2)(-28+2\kappa_1+13\kappa_2) \\ \beta_{2,f}^{(4)} &\simeq 72f_Rf_1\mathcal{H}^2\epsilon^6(-2+\kappa_2)^2(-14+4\kappa_1+5\kappa_2).\end{aligned}\tag{A.3}$$

Coefficients for  $\delta'$  term:

$$\begin{aligned}
\alpha_{1,\text{EH}}^{(1)} &= 432(1+f_R)^{10}\mathcal{H}^3\epsilon^8(-1+\kappa_1)(-2+\kappa_2)^3 \\
\alpha_{1,\text{EH}}^{(2)} &= 2592(1+f_R)^{10}\mathcal{H}^3\epsilon^{10}(-1+\kappa_1)^2(-2+\kappa_2)^3 \\
\alpha_{1,\text{EH}}^{(3)} &= -7776(1+f_R)^{10}\mathcal{H}^3\epsilon^{12}(-1+\kappa_1)^3(-2+\kappa_2)^3 \\
\beta_{1,f}^{(1)} &\simeq 8f_R^4(1+f_R)^6f_1^4\mathcal{H}^3 \\
\beta_{1,f}^{(2)} &\simeq 88f_R^3f_1^3\mathcal{H}^3\epsilon^2(-2+\kappa_2) \\
\beta_{1,f}^{(3)} &\simeq 24f_R^2f_1^2\mathcal{H}^3\epsilon^4(-2+\kappa_2)(-28+2\kappa_1+13\kappa_2) \\
\beta_{1,f}^{(4)} &\simeq 72f_Rf_1\mathcal{H}^3\epsilon^6(-2+\kappa_2)^2(-14+4\kappa_1+5\kappa_2).
\end{aligned} \tag{A.4}$$

Coefficients for  $\delta$  term:

$$\begin{aligned}
\alpha_{0,\text{EH}}^{(1)} &= 432(1+f_R)^{10}\mathcal{H}^4\epsilon^8(-1+\kappa_1)(2\kappa_1-\kappa_2)(-2+\kappa_2)^3 \\
\alpha_{0,\text{EH}}^{(2)} &= 1296(1+f_R)^{10}\mathcal{H}^4\epsilon^{10}(-1+\kappa_1)^2(-1+4\kappa_1-\kappa_2)(-2+\kappa_2)^3 \\
\alpha_{0,\text{EH}}^{(3)} &= 3888(1+f_R)^{10}\mathcal{H}^4\epsilon^{12}(-1+\kappa_1)^2(2\kappa_1^2-\kappa_2)(-2+\kappa_2)^3 \\
\beta_{0,f}^{(1)} &\simeq -\frac{16}{3}f_R^4(1+f_R)^5f_1^4\mathcal{H}^4[2+f_R(2+2f_1-f_2-2\kappa_1)-2\kappa_1] \\
\beta_{0,f}^{(2)} &\simeq 112f_R^3f_1^3\mathcal{H}^4\epsilon^2(-1+\kappa_1)(-2+\kappa_2) \\
\beta_{0,f}^{(3)} &\simeq 48f_R^2f_1^2\mathcal{H}^4\epsilon^4(-1+\kappa_1)(-2+\kappa_2)(-16+2\kappa_1+7\kappa_2) \\
\beta_{0,f}^{(4)} &\simeq 144f_Rf_1\mathcal{H}^4\epsilon^6(-1+\kappa_1)(-2+\kappa_2)^2(-6+4\kappa_1+\kappa_2).
\end{aligned} \tag{A.5}$$

## A.2 Appendix II: $c's$ coefficients

The coefficients appearing in equation (3.59) are explicitly provided here once the condition  $|f_R| \ll 1$  has been imposed.

$$\begin{aligned}
c_4 &= -f_R f_1 [-f_R f_1 k^2 - 3\mathcal{H}^2(-2 + \kappa_2)]^3 \\
c_3 &= -3f_R \mathcal{H} [-f_R f_1 k^2 - 3\mathcal{H}^2(-2 + \kappa_2)] \{f_R^2 f_1^3 k^4 + 6f_2 \mathcal{H}^4(-2 + \kappa_2)^2 + f_1 \mathcal{H}^2(-2 + \kappa_2) \\
&\quad \times [2f_R f_2 k^2 + 3\mathcal{H}^2(-7 + 11\kappa_1 + 4\kappa_2 - 2\kappa_3)] + 2f_R f_1 \mathcal{H}^2 k^2(-6 + 6\kappa_1 + 3\kappa_2 - \kappa_3)\} \\
c_2 &= [-f_R f_1 k^2 - 3\mathcal{H}^2(-2 + \kappa_2)]^2 \\
&\quad \times [f_R^2 f_1^2 k^4 + 5f_R f_1 \mathcal{H}^2 k^2(-2 + \kappa_2) + 6\mathcal{H}^4(-1 + \kappa_1)(-2 + \kappa_2)] \\
c_1 &= [-f_R f_1 k^2 - 3\mathcal{H}^2(-2 + \kappa_2)]^2 \\
&\quad \times [f_R^2 f_1^2 k^4 \mathcal{H} + 5f_R f_1 \mathcal{H}^3 k^2(-2 + \kappa_2) + 6\mathcal{H}^5(-1 + \kappa_1)(-2 + \kappa_2)] \\
c_0 &= \frac{2}{3} \mathcal{H}^2(-1 + \kappa_1) [-f_R f_1 k^2 - 3\mathcal{H}^2(-2 + \kappa_2)]^2 [2f_R^2 f_1^2 k^4 + 9f_R f_1 \mathcal{H}^2 k^2(-2 + \kappa_2) \\
&\quad + 9\mathcal{H}^4(2\kappa_1 - \kappa_2)(-2 + \kappa_2)].
\end{aligned} \tag{A.6}$$



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 [arXiv:0905.1941],  
 Phys. Rev. Lett. **103**, 179001 (2009) [arXiv:0910.1441 [astro-ph.CO]].

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